Spectral Flow and Family Index for Self-Adjoint Elliptic Local Boundary Value Problems on Compact Surfaces

Research Thesis

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The main results of this thesis have been published by the author during the course of the author's doctoral research period:

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Both papers are attached to the thesis, see Appendices B and C. The first paper comprises the results of Section 7 and Parts III, IV, and V. The second paper comprises the results of Section 8 and Part VI.

Contents

Summary

The thesis presents a first step towards a family index theorem for classical elliptic self-adjoint boundary value problems. We address here the simplest non-trivial case of manifolds with boundary, namely the case of two-dimensional manifolds. Over such a manifold, that is, a smooth compact surface with non-empty boundary, we consider first order self-adjoint elliptic differential operators with self-adjoint elliptic local boundary conditions. The first part of our results concerns paths in the space of such boundary value problems connecting two boundary value problems conjugated by a unitary automorphism. We compute the spectral flow for such paths in terms of the topological data over the boundary. In addition, we show that the spectral flow is a universal additive invariant for such paths if the vanishing on paths of invertible operators is required. The second part of our results concerns families of such boundary value problems parametrized by points of an arbitrary compact topological space X. We prove a family index theorem for such families, namely, we compute the $K^1(X)$-valued analytical index of a family in terms of the topological data over the boundary. In addition, we show that the index is a universal additive invariant for such families if the vanishing on families of invertible operators is required.

Preface

An index theory for families of elliptic operators on a closed manifold was developed by Atiyah and Singer in [AS2]. For a family of such operators, parametrized by points of a compact space X, the $K^0(X)$-valued analytical index was computed there in purely topological terms.

An analog of this theory for self-adjoint elliptic operators on closed manifolds was developed by Atiyah, Patodi, and Singer in [APS]; the analytical index of a family in this case takes values in the K^1 group of a base space.

If a family of self-adjoint operators is parametrized by points of a circle, then the $K^1(S^1)$-valued index can be identified with an integer-valued spectral invariant called the spectral flow. This invariant has clear analytical meaning: it counts with signs the number of eigenvalues passing through zero from the start of the path to its end. The theory developed by Atiyah, Patodi, and Singer in [APS] gives also the formula for the spectral flow for a loop of self-adjoint elliptic operators on a closed manifold.

The situation becomes much more complicated if a manifold has non-empty boundary. The integer-valued index of a single boundary value problem was computed by Atiyah and Bott [AB] and Boutet de Monvel [BM]. This result was generalized to the $K^0(X)$-valued analytical index for families of boundary value problems by Melo, Schrohe, and Schick in [MSS].

Manifolds with boundary: self-adjoint case. The case of self-adjoint boundary value problems, however, remains largely open. While Boutet de Monvel's pseudo-differential calculus allows to investigate boundary value problems of different types in a uniform manner, self-adjoint operators seem to lack such a theory. In this case, two different kinds of boundary conditions, global and local, were investigated separately and by different methods.

For Dirac operators on odd-dimensional manifolds with global boundary conditions of Atiya-Patodi-Singer type, Melrose and Piazza computed the odd Chern character of the $K^1(X)$-valued index [MP2], which determines the index up to a torsion. This result is an odd analog of [MP1].

Self-adjoint case: local boundary conditions. For Dirac operators with classical (that is, local) boundary conditions some partial results were obtained in [P1, KN, GL, Yu].

The spectral flow for curves of Dirac operators over a compact surface was computed by the author in [P1] in a very special case, where all the operators have the same symbol and are considered with the same boundary condition. Later the results of [P1] were improved and generalized to manifolds of higher dimension by Katsnelson and Nazaikinskii in [KN] and by Gorokhovsky and Lesch in [GL].

For compact manifolds of arbitrary dimension, Katsnelson and Nazaikinskii expressed the spectral flow of a curve of Dirac operators with local boundary conditions as the (integer-valued) index of the suspension of the curve [KN]. See also [RS] for a similar result in a more general context.

Gorokhovsky and Lesch considered the straight line between the operators $D \otimes \mathrm{Id}$ and $g(D \otimes \mathrm{Id})g^*$, where D is a Dirac operator on an even-dimensional compact manifold M and g is a smooth map from M to the unitary group $\mathcal{U}(\mathbb{C}^n)$. They take the same local boundary condition for all the operators along this line. Using the heat equation approach, they expressed the spectral flow along this line in terms of the spectral flow of boundary Dirac operators [GL].

The result of Gorokhovsky and Lesch was generalized to the higher spectral flow case by Yu. In the general situation, the higher spectral flow is defined for a self-adjoint family parametrized by points of the product $X = Y \times S^1$ provided that the restriction to $Y \times \{pt\}$ has vanishing index in $K^1(Y)$. The higher spectral flow of such a family takes values in $K^0(Y)$ and may be identified with the $K^1(X)$-valued index of the family. Yu considered a Y-parametrized family of Dirac operators on an even-dimensional manifold M, with local boundary conditions, whose $K^1(Y)$-valued index vanishes. From such a family and a map $M \to \mathcal{U}(\mathbb{C}^n)$, he constructed a family of straight lines of Dirac operators, as in [GL]. He expressed the higher spectral flow of such a family in terms of the higher spectral flow of the family of boundary Dirac operators [Yu].

Unfortunately, the methods of [KN, GL, Yu] use essentially the specific nature of Dirac operators and cannot be applied to more general classes of self-adjoint elliptic differential operators.

In this thesis we generalize the previous results in two directions. First, we consider *arbitrary* first order self-adjoint elliptic differential operators, not necessarily of Dirac type. Second, we consider families of such operators parametrized by points of an *arbitrary* compact space X. Our results may be viewed as a first step towards a general family index theorem for classical self-adjoint boundary value problems.

We address here the simplest non-trivial case of manifolds with boundary, namely the case of *two-dimensional* manifolds. We consider first order differential operators on such manifolds with *local*, or classical, boundary conditions (that is, boundary conditions defined by general pseudo-differential operators, in particular boundary conditions of Atiyah-Patodi-Singer type, are not allowed). As it happens, in this setting all the work can be done by topological means only, without using of pseudo-differential calculus.

Spectral flow. Part V of the thesis deals with the simplest non-trivial case of the family index for self-adjoint operators, namely the case of one-dimensional parameter space. In this case the $K^1(S^1)$-valued index can be identified with the spectral flow.

The computation of the spectral flow for paths of first order self-adjoint elliptic operators over a surface is important for some applications in condensed matter physics. For example, the Aharonov-Bohm effect for a single-layer graphene sheet with holes arises if a one-parameter family of Dirac operators has non-zero spectral flow. The varying free term of the Dirac operator corresponds to a varying magnetic field, while the path connecting two gauge equivalent operators corresponds to the situation where magnetic fluxes through holes change by integer numbers in the units of the flux quantum. The spectral flow for such paths of Dirac operators was computed by the author in [P1]. However, some other possible realizations of the Aharonov-Bohm effect in condensed matter physics are described by self-adjoint elliptic operators of non-Dirac type and require different mathematical tools for their investigation.

Part V of the thesis is devoted to the arising mathematical problem, namely computation of the spectral flow for such a family. We compute the spectral flow in terms of the topological data extracted from the corresponding one-parameter family of operators and boundary conditions. In addition, we show that the spectral flow is a universal additive invariant for such a family if the vanishing on families of invertible operators is required.

Family index. In the next part of the thesis, Part VI, we generalize the results of Part V to families of self-adjoint elliptic local boundary value problems on a compact surface parametrized by points of an *arbitrary compact space* X. In this case, the integer-valued spectral flow is replaced by the analytical index taking values in the Abelian group $K^1(X)$.

We prove an *index theorem* for families of such boundary value problems. As we show, the analytical index in our case depends only on the topological data over the boundary. We define the $K^1(X)$-valued topological index in terms of these data and show that the analytical and the topological index coincide.

The second result of Part VI, or rather a collection of results, is *universality of the index* for families of such boundary value problems. We show that the Grothendieck group of homotopy classes of such families modulo the subgroup of invertible families is the K^1-group of the base space, with an isomorphism given by the index. In fact, we prove stronger results, dealing with the semigroup of such families without passing to the Grothendieck group.

List of symbols and abbreviations

Throughout the thesis a "Hilbert space" always means a separable complex Hilbert space of infinite dimension, a "compact space" always means a compact Hausdorff topological space, and a "surface" always means a smooth compact oriented connected surface with non-empty boundary. By the "symbol of a differential operator" we always mean its principal symbol.

$\mathcal{B}(H)$ is the space of bounded linear operators on a Hilbert space H with the norm topology.
$\mathcal{K}(H)$ is the subspace of $\mathcal{B}(H)$ consisting of compact operators.
$\mathcal{U}(H)$ is the subspace of $\mathcal{B}(H)$ consisting of unitary operators.
$\mathcal{U}_K(H)$ is the subspace of $\mathcal{U}(H)$ consisting of unitaries u such that $1 - u$ is a compact operator.

$\mathcal{R}(H)$ denotes the space of regular (that is, closed and densely defined) operators on H equipped with the graph topology.
$\mathcal{R}^{sa}(H)$ is the subspace of $\mathcal{R}(H)$ consisting of self-adjoint operators.
$\mathcal{R}^{sa}_K(H)$ is the subspace of $\mathcal{R}^{sa}(H)$ consisting of operators with compact resolvents.
$\mathcal{R}^{sa}_F(H)$ is the subspace of $\mathcal{R}^{sa}(H)$ consisting of Fredholm operators.

M is a smooth compact Riemannian manifold (in the main part of the thesis it is a surface) with non-empty boundary.
E is a complex Hermitian vector bundle over M.
X is a compact parameter space.

$\mathrm{End}(E)$ is the space of smooth bundle endomorphisms of E.
$U(E)$ is the group of smooth unitary bundle automorphisms of E with the C^1-topology.

Vect_X is the class of all Hermitian vector bundles over X.
Vect^∞_M is the class of all smooth Hermitian vector bundles over M.
$\mathrm{Vect}_{X,M}$ is the class of all locally trivial fiber bundles $\mathcal{E}$ over X with fibers $\mathcal{E}_x \in \mathrm{Vect}^\infty_M$ and the structure group $U(\mathcal{E}_x)$.

$\mathrm{Ell}(E)$ is the space of first order formally self-adjoint elliptic differential operators acting on sections of E.
$\overline{\mathrm{Ell}}(E)$ is the space of pairs (A, L) such that $A \in \mathrm{Ell}(E)$ and L is a self-adjoint elliptic local boundary condition for A.

$\mathrm{sf}(\gamma)$ denotes the spectral flow of a path γ in $\mathcal{R}^{sa}_F(H)$ or in $\overline{\mathrm{Ell}}(E)$.
$\mathrm{ind}(\gamma)$ denotes the family index of a map $\gamma\colon X \to \mathcal{R}^{sa}_K(H)$.
$\mathrm{ind}_a(\gamma)$, respectively $\mathrm{ind}_t(\gamma)$, denotes the analytical, respectively topological, index of a map $\gamma\colon X \to \overline{\mathrm{Ell}}(E)$.

Part I

Introduction

1 Unbounded operators

Let H be a Hilbert space. A closed linear operator A on H is a (not necessarily bounded) linear operator acting from a linear subspace $\operatorname{dom}(A) \subset H$ to H such that its graph is closed in $H \oplus H$. The natural topology on the set of closed operators on H is the so-called graph topology induced by the metric $\delta(A_1, A_2) = \|P_1 - P_2\|$, where P_i denotes the orthogonal projection of $H \oplus H$ onto the graph of A_i.

Denote by $\mathcal{R}(H)$ the space of regular (that is, closed and densely defined) operators on H equipped with the graph topology.

Fredholm operators. The space $\mathcal{R}_F(H)$ of regular Fredholm operators on H, as well as its subspace $\mathcal{R}_K(H)$ consisting of operators with compact resolvents, is a classifying space for the functor K^0 [Jo]. The Fredholm index $\operatorname{ind}(A) = \dim \operatorname{Ker}(A) - \dim \operatorname{Coker}(A)$ of an operator A is constant on the connected components of both $\mathcal{R}_F(H)$ and $\mathcal{R}_K(H)$ and defines the bijection between the set of connected components and the group $\mathbb{Z}$ of integers.

If an operator A is self-adjoint, then its integer-valued index vanishes; the same is true for the $K^0(X)$-valued index of a family of self-adjoint operators parametrized by points of a compact space X. However, self-adjoint operators have another kind of invariant, namely the $K^1(X)$-valued family index, which is identified with the spectral flow if X is a circle.

The spectral flow for self-adjoint unbounded operators. Denote by $\mathcal{R}^{sa}(H)$ the subspace of $\mathcal{R}(H)$ consisting of self-adjoint operators on H and by $\mathcal{R}^{sa}_F(H)$ the subspace of Fredholm self-adjoint operators. The space $\mathcal{R}^{sa}_F(H)$ is path-connected, and its fundamental group is isomorphic to $\mathbb{Z}$. This isomorphism is given by the 1-cocycle on $\mathcal{R}^{sa}_F(H)$ called the spectral flow. Roughly speaking, the spectral flow counts with signs the number of eigenvalues passing through zero from the start of the path to its end (the eigenvalues passing from negative values to positive one are counted with a plus sign, and the eigenvalues passing in the other direction are counted with a minus sign). See [BLP] for a rigorous definition.

The (integer-valued) spectral flow plays the same role for loops of Fredholm self-adjoint operators as the integer-valued index $\dim \operatorname{Ker}(A) - \dim \operatorname{Coker}(A)$ plays for Fredholm operators.

Family index for self-adjoint unbounded operators. Denote by $\mathcal{R}^{sa}_K(H)$ the subspace of $\mathcal{R}(H)$ consisting of self-adjoint operators with compact resolvents. By results of Joachim [Jo], both $\mathcal{R}^{sa}_K(H)$ and $\mathcal{R}^{sa}_F(H)$ are classifying spaces for the functor K^1.

The Cayley transform $A \mapsto \kappa(A) = (A - i)(A + i)^{-1}$ is a continuous embedding of $\mathcal{R}^{sa}(H)$ into the unitary group $\mathcal{U}(H)$. It takes $\mathcal{R}^{sa}_K(H)$ into the subgroup $\mathcal{U}_K(H)$ of $\mathcal{U}(H)$ consisting of unitaries u such that the operator $1 - u$ is compact. Hence $\mathcal{R}^{sa}_K(H)$ can be considered as a subspace of $\mathcal{U}_K(H)$.

As is well known, the group $[X, \mathcal{U}_K(H)]$ of homotopy classes of maps from a compact topological space X to $\mathcal{U}_K(H)$ is naturally isomorphic to $K^1(X)$. We define the *family index* $\operatorname{ind}(\gamma)$ of a continuous map $\gamma\colon X \to \mathcal{R}^{sa}_K(H)$ as the homotopy class of the composition $\kappa \circ \gamma\colon X \to \mathcal{U}_K(H)$ considered as an element of $K^1(X)$,

$$\operatorname{ind}(\gamma) = [\kappa \circ \gamma] \in [X, \mathcal{U}_K(H)] = K^1(X).$$

More generally, this definition works as well for families of regular self-adjoint operators with compact resolvents acting on fibers of a Hilbert bundle over X. See Section 8 for details.

2 Criteria for graph continuity

In this thesis we deal with families of differential operators on a manifold M with boundary. To define the spectral flow or the analytical index for such a family, one needs to ensure that the corresponding family of unbounded operators in $H = L^2(M)$ is graph continuous. For elliptic operators on a closed manifold this is an easy task. However, it is not so clear for boundary value problems on a manifold with boundary. Part III of the thesis is devoted to solving this problem.

First, we deduce some general criteria describing when a family of closed operators in H is graph continuous. Then, in Section 13, we apply these general results to a particular case in hand, namely to differential operators on manifolds with boundary.

On the way we prove a result (Proposition 10.2) which gives an equivalent definition of the gap topology on the space $\operatorname{Gr}(H)$ of all complemented closed linear subspaces of a Banach space H. Namely, the gap topology on $\operatorname{Gr}(H)$ coincides with the quotient topology induced by the map $\operatorname{Proj}(H) \to \operatorname{Gr}(H)$, $P \mapsto \operatorname{Im} P$, where $\operatorname{Proj}(H)$ is the space of all idempotents in $\mathcal{B}(H)$ with the norm topology. The author does not know if this fact was noted before.

3 Elliptic local boundary value problems

Local boundary value problems on a compact manifold. Let M be a smooth compact Riemannian manifold with non-empty boundary ∂M, and let E be a Hermitian vector bundle over M. Denote by E_∂ the restriction of E to ∂M. Let A be a first order formally self-adjoint elliptic differential operator acting on sections of E. We consider only local, or classical, boundary conditions for A, which are given by smooth subbundles of E_∂ (in particular, boundary conditions defined by spectral projections are not allowed). A smooth subbundle L of E_∂ defines the unbounded operator A_L on $L^2(E)$ with the domain

$$\operatorname{dom}(A_L) = \{u \in H^1(E)\colon\ u|_{\partial M} \text{ is a section of } L\},$$

where $H^1(E)$ denotes the first order Sobolev space of sections of E.

The conormal symbol $\sigma(n)$ of A defines the symplectic structure on the fibers of E_∂. For every non-zero cotangent vector $\xi \in T^*_x\partial M$ the operator $\sigma(n_x)^{-1}\sigma(\xi)$ has no eigenvalues on the real axis. The invariant subspaces $E^+(\xi)$ and $E^-(\xi)$ of this operator corresponding to eigenvalues with positive, respectively negative imaginary part are Lagrangian subspaces of E_x.

A local boundary condition L is called elliptic, or Shapiro-Lopatinskii, boundary condition for A if L_x is a complementary subspace for each $E^+(\xi)$, that is,

$$L_x \cap E^+(\xi) = 0 \;\text{ and }\; L_x + E^+(\xi) = E_x \;\text{ for every non-zero } \xi \in T^*_x\partial M. \tag{3.1}$$

If L is elliptic for A, then A_L is a closed operator on $H = L^2(E)$ with compact resolvents. If, in addition, L is a Lagrangian subbundle of E_∂, that is,

$$\sigma(n)L = L^\perp, \tag{3.2}$$

then A_L is self-adjoint.

We denote by $\overline{\mathrm{Ell}}(E)$ the space of all such pairs (A, L) equipped with the C^1-topology on symbols of operators, the C^0-topology on their free terms, and the C^1-topology on boundary conditions. We show in Part III that the natural inclusion

$$\iota\colon \overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_K\left(L^2(E)\right), \quad (A, L) \mapsto A_L, \tag{3.3}$$

is continuous.

Local boundary value problems on a surface. Let now M be an oriented smooth compact *surface*. Then $T^*_x\partial M \setminus \{0\}$ consists of only two rays, so E_∂ can be naturally decomposed into the direct sum $E^+_\partial \oplus E^-_\partial$ of two Lagrangian subbundles. Their fibers can be written as $E^+_x = E^+(\xi)$ and $E^-_x = E^-(\xi)$, where (n, ξ) is a positive oriented frame in T^*_xM.

The identity $E^+(-\xi) = E^-(\xi)$ together with (3.2) allows to simplify ellipticity condition (3.1). Namely, a self-adjoint elliptic local boundary condition for A is a Lagrangian subbundle L of E_∂ satisfying

$$L \cap E^+_\partial = L \cap E^-_\partial = 0.$$

We show in Proposition 15.3 that such subbundles L are in a one-to-one correspondence with self-adjoint bundle automorphisms T of E^-_∂. This correspondence is given by the rule

$$L = \operatorname{Ker} P_T \;\text{ with }\; P_T = P^+\left(1 + i\sigma(n)^{-1}TP^-\right), \tag{3.4}$$

where P^+ denotes the bundle projection of E_∂ onto E^+_∂ along E^-_∂ and $P^- = 1 - P^+$. If A is the Dirac operator, then E^+_∂ and E^-_∂ are mutually orthogonal; in this case L can be written as $L = \left\{u^+ \oplus u^- \in E^+_\partial \oplus E^-_\partial \colon\; i\sigma(n)u^+ = Tu^-\right\}$.

The topological data. We associate with an element $(A, L) \in \overline{\mathrm{Ell}}(E)$ the vector subbundle $F = F(A, L)$ of E_∂^-, whose fibers F_x, $x \in \partial M$, are spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues. As we show in the thesis, in two-dimensional case the vector bundle $F = F(A, L)$ incorporates all the information about (A, L) that we need to compute the spectral flow and the family index.

4 Spectral flow

The natural inclusion (3.3), $\iota \colon \overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_K(H)$ taking (A, L) to A_L is continuous; see Proposition 14.3. Thus the spectral flow $\operatorname{sf}(\gamma)$ is defined for every continuous path $\gamma \colon [0, 1] \to \overline{\mathrm{Ell}}(E)$.

The invariant Ψ. Let $\gamma \colon [0, 1] \to \overline{\mathrm{Ell}}(E)$, $\gamma = (A_t, L_t)$ be a continuous path such that $\gamma(1) = g\gamma(0)$ for some smooth unitary bundle automorphism g of E. With every such pair (γ, g) we associate the vector bundle $\mathcal{F}(\gamma, g)$ over the product $\partial M \times S^1$ as follows. The one-parameter family (F_t) of subbundles $F_t = F(A_t, L_t)$ of E_∂ defines the vector bundle over $\partial M \times [0, 1]$. The condition $\gamma(1) = g\gamma(0)$ implies $F_1 = gF_0$. Gluing F_1 with F_0 twisted by g, that is, identifying $(u, 1)$ with $(gu, 0)$ for every $u \in F_0$, we obtain the vector bundle $\mathcal{F} = \mathcal{F}(\gamma, g)$ over $\partial M \times S^1$.

The product $\partial M \times S^1$ is a disjoint union of tori. The orientation on M induces the orientation on $\partial M \times S^1$. Evaluating the first Chern class of the vector bundle $\mathcal{F}(\gamma, g)$ on the fundamental class of $\partial M \times S^1$, we obtain the integer-valued invariant

$$\Psi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] = \sum_{j=1}^{m} c_1(\mathcal{F}_j)[\partial M_j \times S^1],$$

where ∂M_j, $j = 1 \ldots, m$, are the boundary components and $\mathcal{F}_j$ is the restriction of $\mathcal{F}(\gamma, g)$ to ∂M_j.

The spectral flow formula. The first main result of Part V is the following formula.

Theorem 22.1. *Let $\gamma \colon [0, 1] \to \overline{\mathrm{Ell}}(E)$ be a continuous path such that $\gamma(1) = g\gamma(0)$ for some smooth unitary bundle automorphism g of E. Then the spectral flow of γ can be computed in terms of the topological data over the boundary:*

$$\operatorname{sf}(\gamma) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1] = \Psi(\gamma, g). \tag{4.1}$$

Note that we *do not* require the weak inner unique continuation property for operators $\gamma(t)$. While Dirac operators always have this property, for general first order self-adjoint elliptic operators this is not necessarily so.

Example: conjugation by a scalar function. In one particular case our spectral flow formula (4.1) takes an especially simple form. Let $(A, L) \in \overline{\mathrm{Ell}}(E)$ and let $T = T(A, L)$ be the bundle automorphism of E_∂^- defined by the formula (3.4). The conjugation by a smooth function $g\colon M \to \{z \in \mathbb{C}\colon |z| = 1\}$ preserves the symbol of A, that is, $gAg^{-1} - A = -g^{-1}\sigma_A(dg)$ is a self-adjoint bundle endomorphism. Let $Q\colon [0,1] \to \mathrm{End}^{sa}(E)$ be a one-parameter family of self-adjoint bundle endomorphisms such that $Q_1 = Q_0 - g^{-1}\sigma_A(dg)$.

Theorem 23.1. *The spectral flow of the family* $(A + Q_t,\, L)$ *is equal to* $\sum_{j=1}^{m} \varepsilon_j g_j$*, where* ε_j *is the number of negative eigenvalues of* T *(counting multiplicities) on* ∂M_j *and* g_j *is the degree of the restriction of* g *to* ∂M_j*.*

In Appendix A we apply Theorem 23.1 to Dirac operators on a planar domain. Our aim there is to provide the reader with the simplest and most basic examples. In addition, these examples may be useful for condensed matter physics; one of the possible applications is to the Aharonov-Bohm effect for a single-layer graphene sheet with holes.

Universality of the spectral flow. The second main result of Part V is the universality of the spectral flow for paths in $\overline{\mathrm{Ell}}(E)$ with conjugate ends.

Let $U(E)$ denote the group of smooth unitary bundle automorphisms of E. For $g \in U(E)$ we denote by $\Omega_g\overline{\mathrm{Ell}}(E)$ the space of continuous paths $\gamma\colon [0,1] \to \overline{\mathrm{Ell}}(E)$ such that $\gamma(1) = g\gamma(0)$, equipped with the compact-open topology.

Recall that every complex vector bundle over M is trivial and that $\overline{\mathrm{Ell}}(E)$ is empty for bundles E of odd rank. Denote by $2k_M$ the trivial vector bundle of rank $2k$ over M with the standard Hermitian structure.

Theorem 24.1. *Let* Λ *be a commutative monoid. Suppose that we associate an element* $\Phi(\gamma, g) \in \Lambda$ *with every path* $\gamma \in \Omega_g\overline{\mathrm{Ell}}(2k_M)$ *for every* $k \in \mathbb{N}$ *and every* $g \in U(2k_M)$*. Then the following two conditions are equivalent:*

1. Φ *is homotopy invariant, additive with respect to direct sums, and vanishing on paths of invertible operators.*
2. Φ *has the form* $\Phi(\gamma, g) = c \cdot \mathsf{sf}(\gamma)$ *for some (invertible) constant* $c \in \Lambda$*.*

The homotopy invariance here is understood as the invariance with respect to a change of a path in the space $\Omega_g\overline{\mathrm{Ell}}(E)$ of all paths in $\overline{\mathrm{Ell}}(E)$ with ends conjugated by a *fixed* unitary automorphism g of E. In other words, Φ is constant on path connected components of $\Omega_g\overline{\mathrm{Ell}}(2k_M)$.

By vanishing on paths of invertible operators we mean that Φ vanishes on $\Omega_g\overline{\mathrm{Ell}}^0(2k_M)$ for every k and g, where $\overline{\mathrm{Ell}}^0(E)$ denotes the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that the unbounded operator A_L is invertible (or, what is the same, has no zero eigenvalues).

A similar result holds also for invariants Φ defined only on loops $\gamma \in \Omega\overline{\mathrm{Ell}}(2k_M)$, which is a particular case of Theorem 34.5 from Part VI.

It is known that the spectral flow is a universal homotopy invariant for loops in the space $\mathcal{R}^{sa}_F(H)$, and that the spectral flow is additive with respect to direct sums and vanishes on loops of invertible operators. But the space $\overline{\mathrm{Ell}}(E)$ is only a tiny part of $\mathcal{R}^{sa}_F(L^2(E))$. Universality is usually lost after passing to a subspace, so we cannot expect the spectral flow to be a universal invariant for loops in $\overline{\mathrm{Ell}}(E)$. Indeed, for any given E the map $\mathrm{sf}\colon [S^1, \overline{\mathrm{Ell}}(E)] \to \mathbb{Z}$ is not injective. It is surprising that considering all vector bundles over M together is enough to restore the universality.

Universality of Ψ. The proofs of both the spectral flow formula and the universality of the spectral flow are based upon the following result, universality of Ψ, which we prove in Section 19 using topological means only.

Denote by $\overline{\mathrm{Ell}}^+(E)$, respectively $\overline{\mathrm{Ell}}^-(E)$ the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all (A, L) with positive, respectively negative definite T, where the bundle automorphism $T = T(A, L)$ is defined by formula (3.4).

Theorem 19.3. *Let Λ be a commutative monoid. Suppose that we associate an element $\Phi(\gamma, g) \in \Lambda$ with every path $\gamma \in \Omega_g\overline{\mathrm{Ell}}(2k_M)$ for every $k \in \mathbb{N}$ and $g \in U(2k_M)$. Then the following two conditions are equivalent:*

1. *Φ is homotopy invariant, additive with respect to direct sums, and vanishing on constant loops in $\overline{\mathrm{Ell}}(2k_M)$ and on paths in $\overline{\mathrm{Ell}}^+(2k_M)$, $\overline{\mathrm{Ell}}^-(2k_M)$ for every k.*

2. *Φ has the form $\Phi(\gamma, g) = c \cdot \Psi(\gamma, g)$ for some (invertible) constant $c \in \Lambda$.*

The direction ($2 \Rightarrow 1$) follows immediately from the properties of Ψ. To prove ($1 \Rightarrow 2$), we first notice that if an additive homotopy invariant Φ vanishes on $\Omega_g\overline{\mathrm{Ell}}^+(2k_M)$ and $\Omega\overline{\mathrm{Ell}}^-(2k_M)$ for every k and g, then it depends only on the class of $\mathcal{F}(\gamma, g)$ in $K^0(\partial M \times S^1)$. Next we show that vanishing of Φ on $\Omega_g\overline{\mathrm{Ell}}^-(2k_M)$ cancels the image G^∂ of the homomorphism $K^0(M \times S^1) \to K^0(\partial M \times S^1)$ induced by the embedding $\partial M \times S^1 \hookrightarrow M \times S^1$. Similarly, vanishing of Φ on constant loops cancels the image G^* of the homomorphism $K^0(\partial M) \to K^0(\partial M \times S^1)$ induced by the projection $\partial M \times S^1 \to \partial M$. The subgroup of $K^0(\partial M \times S^1)$ spanned by G^∂ and G^* is the kernel of the surjective homomorphism $\psi\colon K^0(\partial M \times S^1) \to \mathbb{Z}$, which is given by the rule $\psi[V] = c_1(V)[\partial M \times S^1]$ for every vector bundle V over $\partial M \times S^1$. It follows that Φ factors through ψ, that is, $\Phi(\gamma, g) = \vartheta \circ \psi[\mathcal{F}(\gamma, g)] = \vartheta(\Psi(\gamma, g))$ for some monoid homomorphism $\vartheta\colon \mathbb{Z} \to \Lambda$. It remains to take $c = \vartheta(1)$.

Invertible operators. Obviously, every constant loop $\gamma \in \Omega\overline{\mathrm{Ell}}(E)$ is homotopic to a constant loop $\gamma' \in \Omega\overline{\mathrm{Ell}}^0(E)$: one can just add a constant to the corresponding operator.

Denote by $\overline{\mathrm{Dir}}(E)$ the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that A is a Dirac operator which is odd with respect to the chiral decomposition. Two subspaces of $\overline{\mathrm{Dir}}(E)$ play a special role: $\overline{\mathrm{Dir}}^+(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^+(E)$ and $\overline{\mathrm{Dir}}^-(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^-(E)$. It can be easily seen that the unbounded operator A_L is invertible for every $(A, L) \in \overline{\mathrm{Dir}}^+(E)$ or $\overline{\mathrm{Dir}}^-(E)$; see Proposition 21.1 for details. Thus both $\Omega_g\overline{\mathrm{Dir}}^+(E)$ and $\Omega_g\overline{\mathrm{Dir}}^-(E)$ are subspaces of $\Omega_g\overline{\mathrm{Ell}}^0(E)$.

Deformation retraction. We show in Section 20 that the natural embedding $\overline{\mathrm{Dir}}(E) \hookrightarrow \overline{\mathrm{Ell}}(E)$ is a homotopy equivalence. Moreover, we construct a deformation retraction of $\overline{\mathrm{Ell}}(E)$ onto a subspace of $\overline{\mathrm{Dir}}(E)$ preserving $E^-_\partial(A)$ and $F(A,L)$; see Proposition 20.6. Similarly, we construct a deformation retraction of $\Omega_g\overline{\mathrm{Ell}}(E)$ onto a subspace of $\Omega_g\,\overline{\mathrm{Dir}}(E)$ preserving $\mathcal{F}(\gamma, g)$; see Proposition 20.7. Restricting the last retraction to the special subspaces defined above, we obtain a deformation retraction of $\Omega_g\overline{\mathrm{Ell}}^+(E)$ onto a subspace of $\Omega_g\overline{\mathrm{Dir}}^+(E)$ and a deformation retraction of $\Omega_g\overline{\mathrm{Ell}}^-(E)$ onto a subspace of $\Omega_g\overline{\mathrm{Dir}}^-(E)$.

In particular, every path connected component of $\Omega_g\overline{\mathrm{Ell}}^+(E)$, respectively $\Omega_g\overline{\mathrm{Ell}}^-(E)$ contains an element of $\Omega_g\overline{\mathrm{Dir}}^+(E)$, respectively $\Omega_g\overline{\mathrm{Dir}}^-(E)$. It follows that every function Φ satisfying the first condition of Theorem 24.1 should satisfy also the first condition of Theorem 19.3. We use this result to deduce Theorems 22.1 and 24.1 from Theorem 19.3.

Proof of the spectral flow formula. To prove Theorem 22.1, we use the homotopy invariance of the spectral flow, its additivity with respect to direct sums, and vanishing of the spectral flow on paths of invertible operators. In other words, the spectral flow considered as a function $\mathsf{sf}\colon \Omega_g\overline{\mathrm{Ell}}(E) \to \mathbb{Z}$ satisfies the first condition of Theorem 24.1 and thus of Theorem 19.3, with $\Phi = \mathsf{sf}$ and $\Lambda = \mathbb{Z}$. Theorem 19.3 implies that there is an integer constant $c \in \mathbb{Z}$ depending only on M such that $\mathsf{sf}(\gamma) = c \cdot \Psi(\gamma, g)$ for every $\gamma \in \Omega_g\overline{\mathrm{Ell}}(E)$.

It remains to find the factor $c = c_M$. Simple reasoning shows that c_M depends only on the diffeomorphism type of M. We then reduce the computation of c_M to the case of the annulus and compute the factor c_{ann} by direct evaluation. This gives $c_M = c_{\mathrm{ann}} = 1$ for any surface M and completes the proof of Theorem 22.1.

Proof of universality of the spectral flow. The spectral flow, as well as each multiple of it, satisfies the first condition of the theorem. Conversely, suppose that an additive homotopy invariant Φ vanishes on $\Omega_g\overline{\mathrm{Ell}}^0(2k_M)$ for every $k \in \mathbb{N}$ and $g \in U(2k_M)$. Then, as was stated above, Φ satisfies also the first condition of Theorem 19.3. It follows that there is an (invertible) constant $c \in \Lambda$ such that $\Phi(\gamma) = c \cdot \Psi(\gamma, g)$ for every $\gamma \in \Omega_g\overline{\mathrm{Ell}}(2k_M)$, k, and g. Substituting the value of Ψ given by Theorem 22.1, $\Psi(\gamma, g) = \mathsf{sf}(\gamma)$, we obtain the second condition of the theorem.

5 Family index

The analytical index. For a compact space X, the natural inclusion (3.3) associates the analytical index $\mathrm{ind}_a(\gamma) := \mathrm{ind}(\iota \circ \gamma) \in K^1(X)$ with every continuous map $\gamma\colon X \to \overline{\mathrm{Ell}}(E)$.

More generally, let $\mathcal{E}$ be a locally trivial fiber bundle over X, whose fibers $\mathcal{E}_x$ are smooth Hermitian vector bundles over M, and the structure group is the group $U(\mathcal{E}_x)$ of smooth unitary bundle automorphisms of $\mathcal{E}_x$. We denote by $\mathrm{Vect}_{X,M}$ the class of all such bundles $\mathcal{E}$. (Notice that we cannot consider arbitrary vector bundles over

$X \times M$, since we need smoothness with respect to coordinates on M.)

Let $\overline{\mathrm{Ell}}(\mathcal{E})$ be the fiber bundle over X associated with $\mathcal{E}$ and having the fiber $\overline{\mathrm{Ell}}(\mathcal{E}_x)$ over $x \in X$. A section of $\overline{\mathrm{Ell}}(\mathcal{E})$ is a family $x \mapsto (A_x, L_x) \in \overline{\mathrm{Ell}}(\mathcal{E}_x)$ of operators and boundary conditions parametrized by points of X. The natural inclusion $\overline{\mathrm{Ell}}(\mathcal{E}_x) \hookrightarrow \mathcal{R}^{sa}_K(L^2(\mathcal{E}_x))$ allows to define the analytical index for such families. Our first result is the computation of the analytical index in terms of the topological data of a family (A_x, L_x) over ∂M.

The topological index. With each family (A_x, L_x) as above we associate its topological index taking values in $K^1(X)$.

Recall that by Proposition 15.3 self-adjoint elliptic local boundary conditions L for A are in a one-to-one correspondence with self-adjoint bundle automorphisms T of E^-_∂. This correspondence is given by the rule (3.4). With a pair $(A, L) \in \overline{\mathrm{Ell}}(E)$ we associated the subbundle $F = F(A, L)$ of E_∂, whose fibers F_x, $x \in \partial M$ are spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues.

Let $\gamma\colon x \mapsto (A_x, L_x)$ be a section of $\overline{\mathrm{Ell}}(\mathcal{E})$. The family of subbundles $F(A_x, L_x)$ gives rise to the subbundle $\mathcal{F} = \mathcal{F}(\gamma)$ of the restriction $\mathcal{E}_\partial$ of $\mathcal{E}$ to $X \times \partial M$. Let $[\mathcal{F}]$ denotes the class of $\mathcal{F}$ in $K^0(X \times \partial M)$. The second factor ∂M is the disjoint union of boundary components ∂M_j, each of which is a circle. Using the natural homomorphism $K^0(X \times S^1) \to K^1(X)$ and taking the sum over the boundary components, we obtain the homomorphism $\mathrm{Ind}_t\colon K^0(X \times \partial M) \to K^1(X)$. Finally, we define the topological index of γ as the value of Ind_t evaluated on the class $[\mathcal{F}(\gamma)] \in K^0(X \times \partial M)$:

$$\mathrm{ind}_t(\gamma) := \mathrm{Ind}_t\,[\mathcal{F}(\gamma)]) \in K^1(X).$$

Index theorem. The first main result of Part VI is an index theorem.

Theorem 33.2. *The analytical index of γ is equal to its topological index:*

$$\mathrm{ind}_a(\gamma) = \mathrm{ind}_t(\gamma).$$

If the base space X is a circle, then γ is a one-parameter family of operators. In this case, up to the identification $K^1(S^1) \cong \mathbb{Z}$, the analytical index of γ coincides with the spectral flow of γ and the topological index of γ coincides with $c_1(\mathcal{F}(\gamma))[\partial M \times S^1]$. Thus the Spectral Flow Formula (Theorem 22.1) is a particular case of the Index Theorem 33.2.

Properties of the analytical index. The proof of the index theorem is based on the following properties of the analytical index:

(I0) Vanishing on families of invertible operators.

(I1) Homotopy invariance.

(I2) Additivity with respect to direct sum of operators and boundary conditions.

(I3) Functoriality with respect to base changes.

(I4) Multiplicativity with respect to twisting by Hermitian vector bundles over the base space.

(I5) Normalization: the analytical index of a loop $\gamma\colon S^1 \to \overline{\mathrm{Ell}}(E)$ coincides with the spectral flow of γ up to the natural isomorphism $K^1(S^1) \cong \mathbb{Z}$.

Here by an "invertible operator" we mean a boundary value problem (A, L) such that the unbounded operator A_L has no zero eigenvalues (since A_L is self-adjoint, this condition is equivalent to the invertibility of A_L).

These properties follow immediately from the analogous properties of the family index for unbounded operators on a Hilbert space; see Section 8 for details. As it happens, these properties alone are sufficient to prove the index theorem.

Universality of the topological index. To describe all invariants of families of self-adjoint elliptic local boundary problems over M satisfying properties (I0–I5), we note first that the topological index satisfies properties (I1–I4). Property (I0), however, is purely analytical, so its connection with the topological index is not clear a priori. We manage this problem, replacing temporarily (I0) by two topological properties, ($T^{\pm}$) and ($T^{\boxtimes}$), which are stated below.

First, we replace the subspace $\overline{\mathrm{Ell}}^0(E) \subset \overline{\mathrm{Ell}}(E)$ consisting of invertible operators by the two special subspaces $\overline{\mathrm{Ell}}^+(E), \overline{\mathrm{Ell}}^-(E) \subset \overline{\mathrm{Ell}}(E)$, as above. Let $\overline{\mathrm{Ell}}^0(\mathcal{E})$, $\overline{\mathrm{Ell}}^+(\mathcal{E})$, and $\overline{\mathrm{Ell}}^-(\mathcal{E})$ denote the corresponding subbundles of $\overline{\mathrm{Ell}}(\mathcal{E})$. We show that every section of $\overline{\mathrm{Ell}}^+(\mathcal{E})$ or $\overline{\mathrm{Ell}}^-(\mathcal{E})$ is homotopic to a section of $\overline{\mathrm{Ell}}^0(\mathcal{E})$; see Propositions 20.8 and 21.1.

In addition to this, we consider "locally constant" families of operators, that is, sections $1_W \boxtimes (A, L)$ of $\overline{\mathrm{Ell}}(W \boxtimes E)$, where an element $(A, L) \in \overline{\mathrm{Ell}}(E)$ is twisted by a vector bundle W over X. See Section 27 for details. Since every $(A, L) \in \overline{\mathrm{Ell}}(E)$ is connected by a path with an invertible operator, every section of the form $1_W \boxtimes (A, L)$ is homotopic to a section of $\overline{\mathrm{Ell}}^0(W \boxtimes E)$.

Finally, as a substitute for (I0), we take the following two properties:

($T^{\pm}$) Vanishing on sections of $\overline{\mathrm{Ell}}^+(\mathcal{E})$ and $\overline{\mathrm{Ell}}^-(\mathcal{E})$.

($T^{\boxtimes}$) Vanishing on "locally constant" sections.

In Sections 29 and 31 we prove a number of results concerning the universal nature of the topological index; here we show only two of them.

Theorem 29.6. *Let X be a compact space and Λ be a commutative monoid. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ for every $\mathcal{E} \in \mathrm{Vect}_{X,M}$. Then the following two conditions are equivalent:*

1. *Φ satisfies properties ($T^{\pm}$, $T^{\boxtimes}$) and (I1, I2).*

2. *Φ has the form $\Phi(\gamma) = \vartheta(\mathrm{ind}_t(\gamma))$ for some (unique) monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$.*

Theorem 31.1. *Suppose that we associate an element* $\Phi_X(\gamma) \in K^1(X)$ *with every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *for every compact space* X *and every* $\mathcal{E} \in \mathrm{Vect}_{X,M}$*. Then the following two conditions are equivalent:*

1. *The family* $\Phi = (\Phi_X)$ *satisfies properties* ($T^{\pm}$, $T^{\boxtimes}$) *and (I1–I4).*
2. *There is an integer* c *such that* $\Phi = c \cdot \mathrm{ind}_t$.

Note that the factor c here is independent of X.

To deduce Theorem 31.1 from Theorem 29.6, we use the following result about natural self-transformations of the functor K^1.

Proposition 30.1. *Let* ϑ *be a natural self-transformation of the functor* $X \mapsto K^1(X)$ *respecting the* $K^0(\cdot)$*-module structure. Then* ϑ *is multiplication by some integer* c, $\vartheta_X(\mu) = c\mu$ *for every* X *and every* $\mu \in K^1(X)$*. In particular, if* ϑ_{S^1} *is the identity, then* ϑ_X *is the identity for every* X.

The proof of the index theorem. As was noted above, every invariant Φ satisfying properties (I0) and (I1) satisfies also ($T^{\pm}$) and ($T^{\boxtimes}$). Thus Theorem 31.1 implies that every invariant Φ satisfying properties (I0–I4) has the form $\Phi = m \cdot \mathrm{ind}_t$. Applying this to the analytical index, we see that it is an integer multiple of the topological index: $\mathrm{ind}_a = c \cdot \mathrm{ind}_t$ for some integer factor $c = c_M$, which does not depend on X, but may depend on M.

To compute c, it is sufficient to consider the simplest non-trivial base space, namely $X = S^1$, where the analytical index is just the spectral flow. It remains to apply Theorem 22.1 to obtain $c_M = 1$ for any surface M. It follows that the analytical index and the topological index of γ coincide.

Universality of the analytical index. The second main goal of Part VI is universality of the analytical index. We obtain a number of results in this direction in Section 34, combining our index theorem with results of Sections 29 and 31.

Universality for maps. Recall that we denoted by $2k_M$ the trivial vector bundle over M of rank 2k with the standard Hermitian structure.

Theorem 34.4. *Let* $\gamma \colon X \to \overline{\mathrm{Ell}}(2k_M)$, $\gamma' \colon X \to \overline{\mathrm{Ell}}(2k'_M)$ *be continuous maps. Then the following two conditions are equivalent:*

1. $\mathrm{ind}_a(\gamma) = \mathrm{ind}_a(\gamma')$.
2. *There are* $l \in \mathbb{N}$ *and maps* $\beta \colon X \to \overline{\mathrm{Ell}}^0(2(l-k)_M)$, $\beta' \colon X \to \overline{\mathrm{Ell}}^0(2(l-k')_M)$ *such that* $\gamma \oplus \beta$ *and* $\gamma' \oplus \beta'$ *are homotopic as maps from* X *to* $\overline{\mathrm{Ell}}(2l_M)$.

Semigroup of elliptic operators. The disjoint union

$$\overline{\mathrm{Ell}}_M = \coprod_{k \in \mathbb{N}} \overline{\mathrm{Ell}}(2k_M)$$

has the natural structure of a (non-commutative) graded topological semigroup with respect to the direct sum of operators and boundary conditions. The set $[X, \overline{\mathrm{Ell}}_M]$ of homotopy classes of maps from X to $\overline{\mathrm{Ell}}_M$ has the induced semigroup structure. The semigroup $[X, \overline{\mathrm{Ell}}_M]$ is commutative; see Proposition 31.3.

Denote by $\overline{\mathrm{Ell}}^o_M = \coprod_{k\in\mathbb{N}} \overline{\mathrm{Ell}}^o(2k_M)$ the subsemigroup of $\overline{\mathrm{Ell}}_M$ consisting of invertible operators. The inclusion $\overline{\mathrm{Ell}}^o_M \hookrightarrow \overline{\mathrm{Ell}}_M$ induces the homomorphism $[X, \overline{\mathrm{Ell}}^o_M] \to [X, \overline{\mathrm{Ell}}_M]$; we will denote by $[X, \overline{\mathrm{Ell}}_M]^o$ its image. The analytical index is homotopy invariant and vanishes on families of invertible operators, so it factors through $[X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^o$. In other words, there exists a (unique) monoid homomorphism $\kappa_a \colon [X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^o \to K^1(X)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} C(X, \overline{\mathrm{Ell}}_M) & \longrightarrow & [X, \overline{\mathrm{Ell}}_M] & \longrightarrow & [X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^o \\ & \searrow^{\mathrm{ind}_a} & & & \downarrow \kappa_a \\ & & & & K^1(X) \end{array}$$

Theorem 34.6. *The quotient* $[X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^o$ *is an Abelian group isomorphic to* $K^1(X)$*, with an isomorphism given by* κ_a.

The family index is a universal homotopy invariant for maps from X to $\mathcal{R}^{sa}_K(H)$, but the space $\overline{\mathrm{Ell}}(E)$ is only a tiny part of $\mathcal{R}^{sa}_K(L^2(E))$. Universality is usually lost after passing to a subspace, so we cannot expect from the analytical index to be a universal invariant for $\overline{\mathrm{Ell}}(E)$. Indeed, it follows from our index theorem that for any given E the map $\mathrm{ind}_a \colon [X, \overline{\mathrm{Ell}}(E)] \to K^1(X)$ is neither injective nor surjective for general X. It is surprising that universality can be restored by considering all vector bundles over M together.

Universality for families. Denote by $2k_{X,M} \in \mathrm{Vect}_{X,M}$ the trivial bundle over X with the fiber $2k_M$.

Theorem 34.1. *Let* γ_i *be a section of* $\overline{\mathrm{Ell}}(\mathcal{E}_i)$, $i = 1, 2$. *Then the following two conditions are equivalent:*

1. $\mathrm{ind}_a(\gamma_1) = \mathrm{ind}_a(\gamma_2)$.
2. *There are* $k \in \mathbb{N}$, *sections* β^o_i *of* $\overline{\mathrm{Ell}}^o(2k_{X,M})$, *and sections* γ^o_i *of* $\overline{\mathrm{Ell}}^o(\mathcal{E}_i)$ *such that* $\gamma_1 \oplus \gamma^o_2 \oplus \beta^o_1$ *and* $\gamma^o_1 \oplus \gamma_2 \oplus \beta^o_2$ *are homotopic sections of* $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus 2k_{X,M}$.

Let $\mathbb{V}$ be a subclass of $\mathrm{Vect}_{X,M}$ closed under direct sums and containing the trivial bundle $2k_{X,M}$ for every $k \in \mathbb{N}$. In particular, $\mathbb{V}$ may coincide with the whole $\mathrm{Vect}_{X,M}$.

Theorem 34.2. *Let* X *be a compact space and* Λ *be a commutative monoid. Suppose that we associate an element* $\Phi(\gamma) \in \Lambda$ *with every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *for every* $\mathcal{E} \in \mathbb{V}$. *Then the following two conditions are equivalent:*

1. Φ *satisfies properties (I0–I2).*

2. Φ *has the form* $\Phi(\gamma) = \vartheta(\mathrm{ind}_a(\gamma))$ *for some (unique) monoid homomorphism* $\vartheta\colon K^1(X) \to \Lambda$.

Theorem 34.3. *Suppose that we associate an element* $\Phi_X(\gamma) \in K^1(X)$ *with every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *for every compact space* X *and every* $\mathcal{E} \in \mathrm{Vect}_{X,M}$. *Then the following two conditions are equivalent:*

1. *The family* $\Phi = (\Phi_X)$ *satisfies properties (I0–I4).*

2. Φ *has the form* $\Phi_X(\gamma) = m \cdot \mathrm{ind}_a(\gamma)$ *for some integer* m.

Theorems 31.1 and 34.3 are formulated here as statements about the category of compact spaces and continuous maps (maps come into picture due to property (I3)). However, these theorems still remain valid if one replaces this category by the category of finite CW-complexes and continuous maps or by the category of smooth closed manifolds and smooth maps. The choice of such a category comes into the proofs of these theorems only through Proposition 30.1, and we prove this proposition for each of these three categories.

Part II

Unbounded operators and their invariants

6 The space of regular operators

An unbounded operator A on H is a linear operator defined on a subspace $\mathcal{D}$ of H and taking values in H; the subspace $\mathcal{D}$ is called the domain of A and is denoted by $\operatorname{dom}(A)$. An unbounded operator A is called closed if its graph is closed in $H \oplus H$ and densely defined if its domain is dense in H. It is called *regular* if it is closed and densely defined.

Graph topology. Associating with a regular operator on H the orthogonal projection on its graph defines an inclusion of the set of regular operators on H into the space $\operatorname{Proj}(H \oplus H) \subset \mathcal{B}(H \oplus H)$ of projections in $H \oplus H$. Let $\mathcal{R}(H)$ be the set of regular operators on H together with the topology induced from the norm topology on $\operatorname{Proj}(H \oplus H)$ by this inclusion. This topology is usually called the *graph topology, or gap topology*. On the subset $\mathcal{B}(H) \subset \mathcal{R}(H)$ it coincides with the usual norm topology [CL, Addendum, Theorem 1]. So, $\mathcal{B}(H)$ is a subspace of $\mathcal{R}(H)$; it is open and dense in $\mathcal{R}(H)$ [BLP, Proposition 4.1].

A family $\{A_x\}_{x \in X}$ of unbounded operators $A_x \in \mathcal{R}(H)$ defined by a family of differential operators and boundary conditions with continuously varying coefficients leads to a continuous map $X \to \mathcal{R}(H)$; see Propositions 13.2 and 13.3. This property plays a fundamental role in this circle of questions.

Remark 6.1. Another useful topology on the set of regular operators is the *Riesz topology*, induced by the *bounded transform* $A \mapsto A(1 + A^*A)^{-1/2}$ from the norm topology on $\mathcal{B}(H)$. By definition, the bounded transform takes a Riesz continuous family of regular Fredholm operators to a norm continuous family of bounded Fredholm operators, so the index of such a family can be defined in a classical way.

The Riesz topology is well suited for the theory of differential operators on closed manifolds, but, except for several special cases, it is unknown whether families of regular operators on $L^2(E)$ defined by boundary value problems for sections of E are Riesz continuous. Two of such special cases where the Riesz continuity is established are the following: (1) continuous variation of the free term of an operator, with both the higher order part and a boundary condition fixed [Le, Proposition 2.2], and (2) continuous variation of the local boundary condition for a fixed Dirac operator [BR, Theorem 3.1]. However, these results are far from giving Riesz continuity in our framework. In view of this, we use only the graph topology throughout the thesis.

Fredholm operators. A regular operator A is called Fredholm if the range of A is closed and the kernel and the cokernel of A are finite-dimensional. Let $\mathcal{R}_F(H)$ denote the subspace of $\mathcal{R}(H)$ consisting of Fredholm operators.

Operators with compact resolvents. For $A \in \mathcal{R}(H)$, the operator $1 + A^*A \colon \operatorname{dom}(1 + A^*A) \to H$ is regular, bijective, and the inverse operator $(1 + A^*A)^{-1}$ is bounded [Kat, Theorem 3.24]. A regular operator A is said to be an operator *with compact resolvents* if $(1 + A^*A)^{-1}$ is a compact operator. Let $\mathcal{R}_K(H) \subset \mathcal{R}(H)$ denote the subspace of such operators.

An operator with compact resolvent is always Fredholm [Kat, Theorem 6.29], so $\mathcal{R}_K(H)$ is a subspace of $\mathcal{R}_F(H)$.

Self-adjoint regular operators. Recall that the *adjoint operator* of an operator $A \in \mathcal{R}(H)$ is an unbounded operator A^* with the domain

$$\mathrm{dom}(A^*) = \{u \in H\colon \text{ there exists } v \in H \text{ such that } \langle Aw, u\rangle = \langle w, v\rangle \text{ for all } w \in H\}.$$

For $u \in \mathrm{dom}(A^*)$ such an element v is unique and $A^*u = v$ by definition. An operator A is called *self-adjoint* if $A^* = A$ (in particular, $\mathrm{dom}(A^*) = \mathrm{dom}(A)$).

Let $\mathcal{R}^{sa}(H) \subset \mathcal{R}(H)$ denote the subspace of self-adjoint regular operators, $\mathcal{R}^{sa}_F(H)$ denote its subspace consisting of Fredholm operators, and $\mathcal{R}^{sa}_K(H)$ denote the subspace of $\mathcal{R}^{sa}(H)$ consisting of operators with compact resolvents.

For self-adjoint regular operators one can use another, equivalent definition of operators with compact resolvents. Namely, for $A \in \mathcal{R}^{sa}(H)$ the operator $A + i\colon \mathrm{dom}(A) \to H$ is bijective, and the inverse operator $(A + i)^{-1}$ is bounded [Kat, Theorem 3.16]. A self-adjoint regular operator A is an operator *with compact resolvents* if $(A + i)^{-1}$ is a compact operator.

7 Spectral flow for unbounded operators

Fredholm operators and the spectral flow. The space $\mathcal{R}^{sa}_F(H)$ of regular Fredholm self-adjoint operators is path-connected and its fundamental group is isomorphic to $\mathbb{Z}$ [Jo]. This isomorphism is given by the 1-cocycle on $\mathcal{R}^{sa}_F(H)$ called the spectral flow. The definitions of the spectral flow can be found in [Ph] for the case of bounded operators and in [BLP, Le] for the case of unbounded operators.

The case where one or both of the endpoints of the path have zero eigenvalue requires some agreement on the counting procedure. Yet if a path is a loop up to an automorphism of H, the value of the spectral flow is independent of the choice of definition. Since we consider only such paths in this thesis, we do not specify the counting agreement for the case of non-invertible endpoints: any such agreement will suffice.

Properties of the spectral flow. It is well known (see, for example, [P1]) that the spectral flow has a number of nice properties:

(S0) Zero crossing. In the absence of zero crossing the spectral flow vanishes: if γ is a continuous path in $\mathcal{R}^{sa}_F(H)$ such that none of the operators $\gamma(t)$ has zero eigenvalue, then $\mathrm{sf}(\gamma) = 0$.

(S0′) The spectral flow of a constant path vanishes.

(S1) Homotopy invariance. The spectral flow along a continuous path γ in $\mathcal{R}^{sa}_F(H)$ does not change if γ changes continuously in the space of paths in $\mathcal{R}^{sa}_F(H)$ with fixed endpoints (the same as the endpoints of γ).

(S2) Additivity with respect to direct sum. Let H_1, H_2 be separable Hilbert spaces, and let $\gamma_i\colon [a,b] \to \mathcal{R}^{sa}_F(H_i)$ be continuous paths. Then $\mathsf{sf}(\gamma_1 \oplus \gamma_2) = \mathsf{sf}(\gamma_1) + \mathsf{sf}(\gamma_2)$, where $\gamma_1 \oplus \gamma_2\colon [a,b] \to \mathcal{R}^{sa}_F(H_1 \oplus H_2)$ denotes the pointwise direct sum.

(S3) Path additivity. Let γ, γ' be continuous paths in $\mathcal{R}^{sa}_F(H)$ such that the last point of γ is the first point of γ'. Then $\mathsf{sf}(\gamma.\gamma') = \mathsf{sf}(\gamma) + \mathsf{sf}(\gamma')$, where $\gamma.\gamma'$ denotes the concatenation of γ and γ'.

(S4) Conjugacy invariance. Let g be a unitary automorphism of H, and let γ be a continuous path in $\mathcal{R}^{sa}_F(H)$. Then $\mathsf{sf}(\gamma) = \mathsf{sf}(g\gamma g^{-1})$.

Paths with conjugate ends. In the thesis we compute the spectral flow only for paths with conjugate ends (in particular, for loops), so it is convenient to have a special designation for the space of such paths. For a topological space X we denote by ΩX the space of free loops in X with the compact-open topology. Here by a free loop we mean a continuous map from a circle S^1 to X, or, equivalently, a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1)$. If g is a homeomorphism of X, then we denote by $\Omega_g X$ the space of continuous paths $\gamma\colon [0,1] \to X$ such that $\gamma(1) = g\gamma(0)$ equipped with the compact-open topology. We say that paths $\gamma, \gamma' \in \Omega_g X$ are homotopic if they can be connected by a path in $\Omega_g X$.

The group $U(H)$ of unitary automorphisms of H acts on the space $\mathcal{R}^{sa}_F(H)$ by conjugations: $(A, g) \mapsto gAg^{-1}$. We will write $\Omega_g\mathcal{R}^{sa}_F(H)$ for $g \in U(H)$ having in mind this action.

In the proof of Theorem 22.1 we do not use all properties (S0-S4), but only the following small part of them.

Proposition 7.1. *The spectral flow has the following properties.*

(S0^U) ***Zero crossing.*** *Let $\gamma \in \Omega_g\mathcal{R}^{sa}_F(H)$, $g \in U(H)$. Suppose that $\gamma(t)$ has no zero eigenvalue for each $t \in [0,1]$. Then $\mathsf{sf}(\gamma) = 0$.*

(S1^U) ***Homotopy invariance.*** *The spectral flow is constant on path connected components of $\Omega_g\mathcal{R}^{sa}_F(H)$ for each $g \in U(H)$.*

(S2^U) ***Additivity with respect to direct sum.*** *Let $\gamma_i \in \Omega_{g_i}\mathcal{R}^{sa}_F(H_i)$, $g_i \in U(H_i)$, $i = 1, 2$. Then $\mathsf{sf}(\gamma_1 \oplus \gamma_2) = \mathsf{sf}(\gamma_1) + \mathsf{sf}(\gamma_2)$.*

Proof. Properties (S0^U) and (S2^U) are just weaker versions of (S0) and (S2) respectively. To prove (S1^U), we combine (S1), (S3), (S4), and (S0′). Let $\gamma_s(t)$, $s \in [0,1]$ be a homotopy between γ_0 and γ_1. Let the paths $\beta, \beta', \beta''\colon [0,1] \to \mathcal{R}^{sa}_F(H)$ be given by the formulas $\beta(s) = \gamma_s(0)$, $\beta'(t) = \gamma_1(t)$, $\beta''(s) = \gamma_{1-s}(1)$. Then γ_0 is homotopic to $\beta.\beta'.\beta''$ in the space of paths in $\mathcal{R}^{sa}_F(H)$ with the same endpoints as γ_0. Property (S1) implies $\mathsf{sf}(\gamma_0) = \mathsf{sf}(\beta.\beta'.\beta'')$, and by (S3) the last value is equal to $\mathsf{sf}(\beta) + \mathsf{sf}(\beta') + \mathsf{sf}(\beta'')$. Property (S4) implies $\mathsf{sf}(\beta) = \mathsf{sf}(g\beta)$. The path $g\beta$ is just the path β'' passing in the opposite direction, so the concatenation of these two paths is homotopic to the constant path (in the class of paths with fixed endpoints). By (S3), (S1), and (S0′) we have $\mathsf{sf}(g\beta) + \mathsf{sf}(\beta'') = \mathsf{sf}(g\beta.\beta'') = 0$. Taking all this together, we obtain $\mathsf{sf}(\gamma_0) = \mathsf{sf}(\beta') = \mathsf{sf}(\gamma_1)$. □

8 Family index for unbounded operators

In case of bounded operators, the spectral flow of a loop is a particular case of a more general invariant, the family index, which is defined for families of bounded self-adjoint Fredholm operators parametrized by points of a compact topological space. The theory of such families was developed by Atiyah and Singer in [AS1].

In order to deal with *unbounded* self-adjoint operators (in particular, with self-adjoint differential operators) directly, one needs an unbounded analogue of the Atiyah–Singer theory. Cf. [BLP], [BJS], [Jo]. This section is devoted to such an analogue adapted to our framework.

The functor K^1. Let H be a Hilbert space. Denote by $\mathcal{B}(H)$ the space of bounded linear operators $H \to H$ with the norm topology.

The subspace of unitary operators $\mathcal{U}(H) \subset \mathcal{B}(H)$ is a topological group with the multiplication defined by composition. Let $\mathcal{U}_K(H)$ be the subspace of $\mathcal{U}(H)$ consisting of operators u such that $1 - u$ is a compact operator. It is a closed subgroup of $\mathcal{U}(H)$.

The group structure on $\mathcal{U}_K(H)$ induces a (non-commutative) group structure on the space $C(X, \mathcal{U}_K(H))$ of continuous maps from a compact space X to $\mathcal{U}_K(H)$. Passing to the set of connected components of $C(X, \mathcal{U}_K(H))$ defines a group structure on the set $[X, \mathcal{U}_K(H)]$ of homotopy classes of maps from X to $\mathcal{U}_K(H)$. As is well known, the resulting group $[X, \mathcal{U}_K(H)]$ is naturally isomorphic to the classical K^1-theory $K^1(X)$ of X. In particular, it is commutative.

The homotopy type of $\mathcal{R}^{sa}_K(H)$. Booss-Bavnbek, Lesch, and Phillips have shown in [BLP] that the space $\mathcal{R}^{sa}_F(H)$ of Fredholm self-adjoint regular operators is path connected and that the spectral flow defines the surjective homomorphism

$$\pi_1(\mathcal{R}^{sa}_F(H)) \to \mathbb{Z}.$$

They conjectured that $\mathcal{R}^{sa}_F(H)$ is a classifying space for the functor K^1, and this conjecture was proven by Joachim in [Jo]. Along the way he proves (crucially using the results of [BJS]) that $\mathcal{R}^{sa}_K(H)$ is a classifying space for K^1. In our context $\mathcal{R}^{sa}_K(H)$ appears to be a more natural choice of classifying space than $\mathcal{R}^{sa}_F(H)$.

The results of Joachim imply that for a compact space X the set of homotopy classes $[X, \mathcal{R}^{sa}_K(H)]$ of maps $X \to \mathcal{R}^{sa}_K(H)$ is naturally isomorphic to $K^1(X)$. The element of $K^1(X)$ corresponding to a map $\gamma\colon X \to \mathcal{R}^{sa}_K(H)$ deserves to be called the *family index* of γ. At the same time the proofs of the basic properties of this family index depend on a fairly advanced machinery used in [Jo] and [BJS], and the needed properties are not even stated explicitly in these papers.

By this reason we will use another, more elementary, approach to the family index. It is based on the Cayley transform and is a natural development of an idea from [BLP].

The Cayley transform. The Cayley transform of a self-adjoint regular operator A is the unitary operator defined by the formula

$$\kappa(A) = (A - i)(A + i)^{-1} \in \mathcal{U}(H).$$

Proposition 8.1. *The map* $\kappa\colon \mathcal{R}^{sa}(H) \to \mathcal{U}(H)$ *is a continuous embedding. If* A *has compact resolvents, then* $\kappa(A) \in \mathcal{U}_K(H)$*.*

Proof. The first part of the proposition is proven in [BLP, Theorem 1.1]. The second part follows from the identity $1 - \kappa(A) = 2i(A + i)^{-1}$. □

Family index for maps. Recall that $K^1(X) = [X, \mathcal{U}_K(H)]$. The Cayley transform

$$\kappa\colon \mathcal{R}^{sa}_K(H) \to \mathcal{U}_K(H) \tag{8.1}$$

induces the map

$$\kappa_*\colon [X, \mathcal{R}^{sa}_K(H)] \to [X, \mathcal{U}_K(H)] = K^1(X). \tag{8.2}$$

It is proved in [P6] that (8.1) is a weak homotopy equivalence and that the induced map (8.2) is bijective for every compact space X. This motivates our definition of the family index. Let $\gamma\colon X \to \mathcal{R}^{sa}_K(H)$ be a continuous map. We define the *family index* $\operatorname{ind}(\gamma)$ of γ as the homotopy class of the composition $\kappa \circ \gamma\colon X \to \mathcal{U}_K(H)$ considered as an element of $K^1(X)$. In other terms,

$$\operatorname{ind}(\gamma) = [\kappa \circ \gamma] \in [X, \mathcal{U}_K(H)] = K^1(X). \tag{8.3}$$

One can also define in this way the family index of maps $X \to \mathcal{R}^{sa}_F(H)$. But for our purposes it is sufficient to consider only maps $X \to \mathcal{R}^{sa}_K(H)$.

Families of regular operators. More generally, one can consider X-parametrized families $(T_x)_{x\in X}$ of regular operators acting on a X-parametrized family of Hilbert spaces $(\mathcal{H}_x)_{x\in X}$, i.e. on the fibers of a Hilbert bundle $\mathcal{H} \to X$.

In more details, let $\mathcal{H} \to X$ be a Hilbert bundle, that is, a locally trivial fiber bundle over X with a fiber H and the structure group $\mathcal{U}(H)$ (we consider only Hilbert bundles with separable fibers of infinite dimension). Recall that the group $\mathcal{U}(H)$ continuously acts on the space $\mathcal{R}(H)$ by conjugations: $(T, g) \mapsto gTg^{-1}$. The subspace $\mathcal{R}^{sa}_K(H)$ is invariant under this action. This allows to associate with $\mathcal{H}$ the fiber bundle $\mathcal{R}^{sa}_K(\mathcal{H})$ having $\mathcal{R}^{sa}_K(\mathcal{H}_x)$ as the fiber over $x \in X$. We equip the set $\Gamma\mathcal{R}^{sa}_K(\mathcal{H})$ of sections of $\mathcal{R}^{sa}_K(\mathcal{H})$ with the compact-open topology.

By the Kuiper theorem [Ku], the unitary group $\mathcal{U}(H)$ is contractible. Therefore, every Hilbert bundle $\mathcal{H}$ is trivial and a trivialization is unique up to homotopy. The choice of a trivialization identifies sections of $\mathcal{R}^{sa}_K(\mathcal{H})$ with maps from X to $\mathcal{R}^{sa}_K(H)$. The *family index* of a section of $\mathcal{R}^{sa}_K(\mathcal{H})$ is defined as the index of the corresponding map $X \to \mathcal{R}^{sa}_K(H)$. This definition does not depend on the choice of trivialization.

Connection with topological K-theory. Let again X be a compact space. The group $K^1(X)$ may also be defined as the direct limit $\lim_{n\to\infty}[X, \mathcal{U}(\mathbb{C}^n)]$ with respect to the sequence of embeddings

$$\mathcal{U}(\mathbb{C}^1) \hookrightarrow \mathcal{U}(\mathbb{C}^2) \ldots \hookrightarrow \mathcal{U}(\mathbb{C}^n) \hookrightarrow \mathcal{U}(\mathbb{C}^{n+1}) \hookrightarrow \ldots \tag{8.4}$$

given by the rule $u \mapsto u \oplus 1$.

The choice of an orthonormal basis in H allows to identify (8.4) with a sequence of subgroups of $\mathcal{U}_K(H)$. By results of Palais [Pa], the resulting inclusion $j\colon \mathcal{U}_\infty \to \mathcal{U}_K(H)$ of the direct limit $\mathcal{U}_\infty = \lim_{n\to\infty} \mathcal{U}(\mathbb{C}^n)$ is a homotopy equivalence. In particular, every continuous map $u\colon X \to \mathcal{U}_K(H)$ is homotopic to a composition $j \circ \nu$ for some map $\nu\colon X \to \mathcal{U}_\infty$. Since X is compact, every map from X to $\mathcal{U}_\infty$ takes values in some $\mathcal{U}(\mathbb{C}^n)$. Therefore, every map $u\colon X \to \mathcal{U}_K(H)$ is homotopic to a map $X \to \mathcal{U}(\mathbb{C}^n) \subset \mathcal{U}_K(H)$ for sufficiently large n. Similarly, if two maps $u, \nu\colon X \to \mathcal{U}(\mathbb{C}^n)$ are homotopic as maps to $\mathcal{U}_K(H)$, then they are homotopic as maps to $\mathcal{U}(\mathbb{C}^m)$ for some $m \geqslant n$.

The definition of addition in the group $[X, \mathcal{U}_K(H)]$ given in the beginning of the section uses the multiplicative structure of $\mathcal{U}_K(H)$. The standard definition of addition in $\lim_n[X, \mathcal{U}(\mathbb{C}^n)]$ associates with a pair of maps $u, \nu\colon X \to \mathcal{U}(\mathbb{C}^n)$ the direct sum $u \oplus \nu\colon X \to \mathcal{U}(\mathbb{C}^{2n})$, so that $[u] + [\nu] = [u \oplus \nu] \in K^1(X)$. These two definitions are equivalent, since $u \oplus \nu$ and $u\nu \oplus 1$ are homotopic.

Let now $\mathcal{H}$ be a Hilbert bundle over X with a fiber H. The structure group $\mathcal{U}(H)$ of $\mathcal{H}$ acts on $\mathcal{R}^{sa}_K(H)$ and $\mathcal{U}_K(H)$ by conjugations. The Cayley transform $\kappa\colon \mathcal{R}^{sa}_K(H) \to \mathcal{U}_K(H)$ is equivariant with respect to this action. Therefore, κ can be applied point-wise to sections of $\mathcal{R}^{sa}_K(\mathcal{H})$. For a section γ of $\mathcal{R}^{sa}_K(\mathcal{H})$, the Cayley transform $u = \kappa(\gamma)$ is a section of $\mathcal{U}_K(\mathcal{H})$.

Choose a trivialization $J\colon \mathcal{H} \to H_X$, where H_X denotes the trivial Hilbert bundle $H \times X \to X$. The composition $u' = J \circ u$ is a map from X to $\mathcal{U}_K(H)$ and thus is homotopic to $\nu' \oplus 1$ for some map $\nu'\colon X \to \mathcal{U}(\mathbb{C}^n) \subset \mathcal{U}_K(H)$. The classes of u and ν' in $K^1(X)$ coincide. Returning back to $\mathcal{H}$ by applying J^{-1}, we obtain a trivial subbundle E of $\mathcal{H}$ of finite rank and a unitary bundle automorphism ν of E such that the sections u and $\nu \oplus 1$ of $\mathcal{U}_K(\mathcal{H}) = \mathcal{U}_K(E \oplus E^\perp)$ are homotopic.

Conversely, let E be a (not necessarily trivial) vector bundle over X. A bundle automorphism ν of E defines an element $[\nu] \in K^1(X)$ as follows. Lift E to the product $X \times [0, 1]$ and identify the restrictions of E to $X \times \{0\}$ and $X \times \{1\}$ twisting the first one by ν. This constructions gives a vector bundle over $X \times S^1$ which we denote by E_ν. Let $[E_\nu]$ denotes the class of E_ν in $K^0(X \times S^1)$. The group $K^0(X \times S^1)$ is naturally isomorphic to the direct sum $K^0(X) \oplus K^1(X)$; denote by

$$\alpha\colon K^0(X \times S^1) \to K^1(X) \tag{8.5}$$

the projection to the second summand. Then $[\nu] = \alpha[E_\nu] \in K^1(X)$. If E is a subbundle of a Hilbert bundle $\mathcal{H}$ and $u = \nu \oplus 1$ is a section of $\mathcal{U}_K(\mathcal{H})$, then $[u] = [\nu] \in K^1(X)$.

Twisting. One of the key properties of the index that we need in the thesis is its multiplicativity with respect to twisting by vector bundles.

Let V be a finite-dimensional complex vector space equipped with a Hermitian structure. An unbounded operator A on H can be twisted by V, resulting in the unbounded operator $1_V \otimes A$ on $V \otimes H$ with the domain $\operatorname{dom}(1_V \otimes A) = V \otimes \operatorname{dom}(A)$. If an isomorphism $V \cong \mathbb{C}^d$ is chosen, then $V \otimes H$ can be identified with the direct sum of d copies of H and $1_V \otimes A$ can be identified with the direct sum of d copies of A. If $A \in \mathcal{R}^{sa}_K(H)$, then $1_V \otimes A \in \mathcal{R}^{sa}_K(V \otimes H)$.

Let now W be a finite rank Hermitian vector bundle over X. A Hilbert bundle $\mathcal{H}$ over X can be twisted by W, giving rise to another Hilbert bundle $W \otimes \mathcal{H}$ over X with the fiber $(W \otimes \mathcal{H})_x = W_x \otimes \mathcal{H}_x$ over $x \in X$. A section γ of $\mathcal{R}^{sa}_K(\mathcal{H})$ can be twisted by W, resulting in the section $1_W \otimes \gamma$ of $\mathcal{R}^{sa}_K(W \otimes \mathcal{H})$ such that $(1_W \otimes \gamma)(x) = 1_{W_x} \otimes \gamma(x)$. Since the Cayley transform is additive with respect to direct sums and equivariant with respect to conjugation by unitaries, $\kappa(1_W \otimes \gamma) = 1_W \otimes \kappa(\gamma)$.

Choose a subbundle $E \subset \mathcal{H}$ of finite rank and a unitary bundle automorphism ν of E such that the sections $\kappa(\gamma)$ and $\nu \oplus 1$ of $\mathcal{U}_K(\mathcal{H})$ are homotopic. Then the sections $1_W \otimes \kappa(\gamma)$ and $(1_W \otimes \nu) \oplus 1_{W \otimes E^\perp}$ of $\mathcal{U}_K(W \otimes \mathcal{H})$ are also homotopic. The vector bundle $(W \otimes E)_{1_W \otimes \nu}$ is isomorphic to $p^*W \otimes E_\nu$, where p denotes the projection $X \times S^1 \to X$. Since (8.5) is a homomorphism of $K^0(X)$-modules, we get

$$[1_W \otimes \nu] = \alpha[(W \otimes E)_{1_W \otimes \nu}] = \alpha(p^*[W] \cdot [E_\nu]) = [W] \cdot \alpha[E_\nu] = [W] \cdot [\nu] \in K^1(X).$$

It follows that

$$\operatorname{ind}(1_W \otimes \gamma) = [1_W \otimes \kappa(\gamma)] = [1_W \otimes \nu] = [W] \cdot [\nu] = [W] \cdot [\kappa(\gamma)] = [W] \cdot \operatorname{ind}(\gamma) \in K^1(X).$$

Properties of the family index. In fact, we do not need an exact definition of the family index to prove the main results of the thesis. All we need are the following properties of the index.

Proposition 8.2. *The family index has the following properties for every compact spaces X, Y and Hilbert bundles $\mathcal{H}$, $\mathcal{H}'$ over X.*

- *(I0) Vanishing. The index of a family of invertible operators vanishes.*
- *(I1) Homotopy invariance. If γ_0 and γ_1 are homotopic sections of $\mathcal{R}^{sa}_K(\mathcal{H})$, then* $\operatorname{ind}(\gamma_0) = \operatorname{ind}(\gamma_1)$.
- *(I2) Additivity.* $\operatorname{ind}(\gamma_0 \oplus \gamma_1) = \operatorname{ind}(\gamma_0) + \operatorname{ind}(\gamma_1)$ *for every sections γ_i of $\mathcal{R}^{sa}_K(\mathcal{H}_i)$, $i = 0, 1$.*
- *(I3) Functoriality. Let $f\colon Y \to X$ be a continuous map and γ be a section of $\mathcal{R}^{sa}_K(\mathcal{H})$. Then* $\operatorname{ind}(f^*\gamma) = f^* \operatorname{ind}(\gamma) \in K^1(Y)$, *where $f^*\gamma = \gamma \circ f$ is the section of $\mathcal{R}^{sa}_K(f^*\mathcal{H})$.*
- *(I4) Twisting.* $\operatorname{ind}(1_W \otimes \gamma) = [W] \cdot \operatorname{ind}(\gamma)$ *for every section γ of $\mathcal{R}^{sa}_K(\mathcal{H})$ and every Hermitian vector bundle W over X, where $[W]$ denotes the class of W in $K^0(X)$.*
- *(I5) Normalization. For a loop $\gamma\colon S^1 \to \mathcal{R}^{sa}_K(H)$, the index of γ coincides with the spectral flow of γ up to the natural isomorphism $K^1(S^1) \cong \mathbb{Z}$.*

(I6) *Conjugacy invariance. The index of a section of* $\mathcal{R}^{sa}_K(\mathcal{H})$ *is invariant with respect to the conjugation by a unitary bundle automorphism of* $\mathcal{H}$. *In other words,* $\operatorname{ind}(u\gamma u^*) = \operatorname{ind}(\gamma)$ *for every section* γ *of* $\mathcal{R}^{sa}_K(\mathcal{H})$ *and every section* u *of* $\mathcal{U}(\mathcal{H})$.

Proof. (I1) and (I3) follows immediately from the definition of the index. (I4) is proven in the previous subsection.

(I0). The Cayley transform takes the subspace of $\mathcal{R}^{sa}_K(H)$ consisting of invertible operators to the subspace $\mathcal{U}^0_K(H) = \{u \in \mathcal{U}_K(H) \colon u + 1 \text{ is invertible}\}$ of $\mathcal{U}_K(H)$. The space $\mathcal{U}^0_K(H)$ is contractible, with the contraction given by the formula $h_t(u) = \exp(t \log(u))$, where $\log \colon \mathcal{U}(\mathbb{C}) \setminus \{-1\} \to i(-\pi, \pi) \subset i\mathbb{R}$ is a branch of the natural logarithm. Therefore, for every section γ of $\mathcal{R}^{sa}_K(\mathcal{H})$ consisting of invertible operators the composition $\kappa \circ \gamma$ is a section of $\mathcal{U}^0_K(\mathcal{H})$ homotopic to the identity section, so $\operatorname{ind}(\gamma) = [\kappa \circ \gamma] = 0$.

(I2). Let $u_i = \kappa(\gamma_i)$. The Cayley transform is additive with respect to direct sums, so $\kappa(\gamma_0 \oplus \gamma_1) = \kappa(\gamma_0) \oplus \kappa(\gamma_1)$. Let E_i be a trivial subbundle of $\mathcal{H}_i$ of finite rank and ν_i be a unitary bundle automorphism of E_i such that the sections $\kappa(\gamma_i)$ and $\nu_i \oplus 1$ of $\mathcal{U}_K(\mathcal{H}_i)$ are homotopic. Then $\kappa(\gamma_0) \oplus \kappa(\gamma_1)$ and $(\nu_0 \oplus \nu_1) \oplus 1$ are also homotopic, and $\operatorname{ind}(\gamma_0 \oplus \gamma_1) = [\nu_0 \oplus \nu_1] = [\nu_0] + [\nu_1] = \operatorname{ind}(\gamma_0) + \operatorname{ind}(\gamma_1)$.

(I5) follows from [BLP, Proposition 2.17].

(I6). Since the unitary group of a Hilbert space is contractible, there is a homotopy $(u_t)_{t \in [0,1]}$ connecting $u_0 = 1$ and $u_1 = u$. It induces the homotopy $\nu_t = u_t \nu u_t^*$ connecting the sections $\nu = \kappa(\gamma)$ and $u\nu u^*$ of $\mathcal{U}_K(\mathcal{H})$. Therefore, $\operatorname{ind}(u\gamma u^*) = [u\nu u^*] = [\nu] = \operatorname{ind}(\gamma) \in K^1(X)$. □

Part III

Criteria for graph continuity

This part presents some general criteria describing when a family of closed operators (in particular, differential operators on a manifold with boundary) is graph continuous.

Section 13 is devoted to a particular case of these results, namely differential and pseudo-differential operators on a manifold with boundary. We use the results of Section 13 in the main part of the thesis for two purposes. First, Proposition 14.3 arises as a particular case of Proposition 13.3. Second, Proposition 13.2 and Lemma 13.4 provide the continuity of the family of global boundary value problems used in the proof of Lemma 22.5.

Some of the results of sections 11 and 12, though in a different form and with different proofs, are contained in the Appendix to the recent paper of Booss-Bavnbek and Zhu [BZ]. In particular, our Proposition 11.1 is a corollary of [BZ, Proposition A.6.2] and our Proposition 12.1 is a special case of [BZ, Corollary A.6.4]. However, the statements of our Proposition 11.1 and Proposition 12.1 better meet our needs. For Hilbert spaces, our proofs have the advantage of not using elaborated estimates and inequalities. We also add the more general case of Banach spaces with the purpose of better matching the results of [BZ], though we use only Hilbert spaces in the remaining parts of the thesis.

It is worth noticing that our Proposition 10.2 gives an equivalent definition of the gap topology on the space $\mathrm{Gr}(H)$ of all complemented closed linear subspaces of a Banach space H. Namely, the gap topology on $\mathrm{Gr}(H)$ coincides with the quotient topology induced by the map $\mathrm{Proj}(H) \to \mathrm{Gr}(H)$, $P \mapsto \operatorname{Im} P$, where $\mathrm{Proj}(H)$ is the space of all idempotents in $\mathcal{B}(H)$ with the norm topology. The author does not know if this fact was noted before.

9 Complementary pairs of subspaces

Subspaces of a Banach space. Let H be a Banach space. Denote by $\mathcal{B}(H)$ the space of all bounded linear operators on H with the norm topology. Denote by $\mathrm{Proj}(H)$ the subspace of $\mathcal{B}(H)$ consisting of all idempotents.

A closed subspace $L \subset H$ is called complemented if there is another closed subspace $M \subset H$ such that $L \cap M = 0$, $L + M = H$; such a pair (L, M) is called a complementary pair, and M is called a complementary subspace to L. Equivalently, $L \subset H$ is complemented if it is the image of some $P \in \mathrm{Proj}(H)$; (L, M) is a complementary pair if it is equal to $(\operatorname{Im} P, \operatorname{Ker} P)$ for some $P \in \mathrm{Proj}(H)$.

We denote by $\mathrm{Gr}(H)$ the set of all complemented closed linear subspaces of H, and by $\mathrm{Gr}^{(2)}(H)$ the set of all complementary pairs of subspaces of H. We will also write $\mathrm{Gr}^2(H)$ instead of $\mathrm{Gr}(H)^2$ for convenience.

For $(L, M) \in \mathrm{Gr}^{(2)}(H)$ we denote by $P_{L,M}$ the projection of H onto L along M. For $M \in \mathrm{Gr}(H)$ denote by $\mathrm{Gr}^M(H) = \left\{ L \in \mathrm{Gr}(H) \colon (L, M) \in \mathrm{Gr}^{(2)}(H) \right\}$ the set of all complementary subspaces to M.

Proposition 9.1. *Let* H *be a Banach space and* $P, Q \in \mathrm{Proj}(H)$*. Then the following two conditions are equivalent:*

1. *Both* $(\mathrm{Im}\,P, \mathrm{Im}\,Q)$ *and* $(\mathrm{Ker}\,P, \mathrm{Ker}\,Q)$ *are complementary pairs of subspaces.*
2. $P - Q$ *is invertible.*

If this is the case, then for the projection S *on* $\mathrm{Im}\,P$ *along* $\mathrm{Im}\,Q$ *and the projection* T *on* $\mathrm{Ker}\,P$ *along* $\mathrm{Ker}\,Q$ *we have:*

$$S = P(P-Q)^{-1}, \quad T = (P-1)(P-Q)^{-1}, \quad (P-Q)^{-1} = S - T, \tag{9.1}$$

and $P + Q = (2S-1)(P-Q)$ *is also invertible.*

Proof. $(1 \Rightarrow 2)$ Let $(\mathrm{Im}\,P, \mathrm{Im}\,Q), (\mathrm{Ker}\,P, \mathrm{Ker}\,Q) \in \mathrm{Gr}^{(2)}(H)$. Denote by S, T the elements of $\mathrm{Proj}(H)$ corresponding to these two pairs of complementary subspaces. Using the identities $SP = P$, $TQ = T$, $SQ = 0$, and $(1-T)(1-P) = 0$, we obtain

$$(S-T)(P-Q) = T + P - TP = 1 - (1-T)(1-P) = 1.$$

Similarly, we have

$$(P-Q)(S-T) = Q + S - QS = 1 - (1-Q)(1-S) = 1.$$

Therefore, $P - Q$ is invertible with $S - T$ the inverse operator.

$(2 \Rightarrow 1)$ Let $P - Q$ be invertible. It vanishes on the intersections $\mathrm{Im}\,P \cap \mathrm{Im}\,Q$ and $\mathrm{Ker}\,P \cap \mathrm{Ker}\,Q$, so these intersections are trivial. Consider the operators $S = P(P-Q)^{-1}$ and $S' = -Q(P-Q)^{-1}$. We have $\mathrm{Im}\,S = \mathrm{Im}\,P$, $\mathrm{Im}\,S' = \mathrm{Im}\,Q$, and $S + S' = 1$, so $\mathrm{Im}\,P + \mathrm{Im}\,Q = H$. Similarly, consider the operators $T = (P-1)(P-Q)^{-1}$ and $T' = (1-Q)(P-Q)^{-1}$. We have $\mathrm{Im}\,T = \mathrm{Ker}\,P$, $\mathrm{Im}\,T' = \mathrm{Ker}\,Q$, and $T + T' = 1$, so $\mathrm{Ker}\,P + \mathrm{Ker}\,Q = H$. All four subspaces $\mathrm{Im}\,P$, $\mathrm{Im}\,Q$, $\mathrm{Ker}\,P$, $\mathrm{Ker}\,Q$ are closed. Therefore, both $(\mathrm{Im}\,P, \mathrm{Im}\,Q)$ and $(\mathrm{Ker}\,P, \mathrm{Ker}\,Q)$ lie in $\mathrm{Gr}^{(2)}(H)$.

The first equality of (9.1) implies $(2S-1)(P-Q) = 2P - (P-Q) = P + Q$. Note that invertibility of $P + Q$ implies $\mathrm{Im}\,P + \mathrm{Im}\,Q = H$, but *does not* imply $\mathrm{Im}\,P \cap \mathrm{Im}\,Q = 0$. □

Subspaces of a Hilbert space. If H is a Hilbert space, then each closed subspace of H is complemented, so $\mathrm{Gr}(H)$ is the set of all closed subspaces of H. The map $\mathrm{Im}\colon \mathrm{Proj}(H) \to \mathrm{Gr}(H)$ has a natural section taking a closed subspace $L \subset H$ to the orthogonal projection P_L of H onto L. Applying Proposition 9.1, we obtain the following result.

Proposition 9.2. *Let* H *be a Hilbert space. Then the following statements hold:*

1. *The pair* (L, M) *of closed subspaces of* H *is complementary if and only if* $P_L - P_M$ *is invertible. If this is the case, then*

$$P_{L,M} = P_L(P_L - P_M)^{-1}. \tag{9.2}$$

2. *Let* $P \in \mathrm{Proj}(H)$. *Then the operator* $P + P^* - 1$ *is invertible, and the orthogonal projection on the image of* P *is given by the formula*

$$P^{\mathrm{ort}} = P(P + P^* - 1)^{-1}. \tag{9.3}$$

Proof. 1. If $(L, M) \in \mathrm{Gr}^{(2)}(H)$, then also $(L^{\perp}, M^{\perp}) \in \mathrm{Gr}^{(2)}(H)$. Applying Proposition 9.1 to the pair of orthogonal projections P_L and P_M, we obtain the first claim of the Corollary.

2. $1 - P^*$ is the projection on $(\mathrm{Im}\, P)^{\perp}$ along $(\mathrm{Ker}\, P)^{\perp}$. Applying Proposition 9.1 to the pair of projections P and $1 - P^*$, we see that $P + P^* - 1 = P - (1 - P^*)$ is invertible and $P(P + P^* - 1)^{-1}$ is the projection on $\mathrm{Im}\, P$ along $(\mathrm{Im}\, P)^{\perp}$. □

10 The gap topology on $\mathrm{Gr}(H)$

For a *Hilbert space* H the map $L \mapsto P_L$ given by the orthogonal projection allows to identify $\mathrm{Gr}(H)$ with the subspace $\mathrm{Proj}^{\mathrm{ort}}(H) \subset \mathrm{Proj}(H)$ of orthogonal projections in H. The gap topology on $\mathrm{Gr}(H)$ is induced by the norm topology on $\mathrm{Proj}(H) \subset \mathcal{B}(H)$.

For a *Banach space* H there is no natural section $\mathrm{Gr}(H) \to \mathrm{Proj}(H)$, so the definition of the gap topology on $\mathrm{Gr}(H)$ is slightly more complicated in this case. Usually the gap topology on $\mathrm{Gr}(H)$ is defined as the topology induced by the gap metric

$$\hat{\delta}(L_1, L_2) = \max_{i \neq j} \left\{ \sup \left\{ \mathrm{dist}(u, L_j) \colon u \in L_i, \|u\| = 1 \right\} \right\}, \tag{10.1}$$

$$\hat{\delta}(0, 0) = 0, \quad \hat{\delta}(0, L) = 1 \text{ for } L \neq 0.$$

For a Hilbert space H these two definitions of the gap topology coincide.

Proposition 10.2 below gives an equivalent definition of the gap topology on the Grassmanian of a Banach space in terms of projections, resembling the definition of the gap topology for Hilbert spaces.

The gap topology on $\mathrm{Gr}(H)$ induces the topology on $\mathrm{Gr}^2(H)$ and on its subspace $\mathrm{Gr}^{(2)}(H)$.

Proposition 10.1. *Let* H *be a Banach space. Then the following statements hold:*

1. *The map* $\mathrm{Im}\colon \mathrm{Proj}(H) \to \mathrm{Gr}(H)$ *is continuous.*
2. *The map* $\varphi\colon \mathrm{Proj}(H) \to \mathrm{Gr}^{(2)}(H)$ *taking* $P \in \mathrm{Proj}(H)$ *to* $(\mathrm{Im}\, P, \mathrm{Ker}\, P) \in \mathrm{Gr}^{(2)}(H)$ *is a homeomorphism.*
3. $\mathrm{Gr}^{(2)}(H)$ *is open in* $\mathrm{Gr}^2(H)$.

We first give the proof in the case of a Hilbert space H, because it is simpler and because we need only this case in the main part of the thesis as well as in the proofs of all the results below in the context of Hilbert spaces. After proving the "Hilbert case" we give the proof of the general "Banach case".

Proof. 1. *Suppose first that* H *is a Hilbert space.* The map $\operatorname{Im}\colon \operatorname{Proj}(H) \to \operatorname{Gr}(H)$ is continuous. Indeed, it is the composition of the two maps $\operatorname{Proj}(H) \to \operatorname{Proj}^{\mathrm{ort}}(H)$ and $\operatorname{Im}\colon \operatorname{Proj}^{\mathrm{ort}}(H) \to \operatorname{Gr}(H)$, where the first map is given by formula (9.3) and $\operatorname{Proj}^{\mathrm{ort}}(H)$ is the subspace of $\operatorname{Proj}(H)$ consisting of orthogonal projections. The first map is continuous and the second map is an isometry, so their composition is also continuous.

The conjugation by the involution $P \mapsto 1 - P$ takes the map $\operatorname{Im}\colon \operatorname{Proj}(H) \to \operatorname{Gr}(H)$ to the map $\operatorname{Ker}\colon \operatorname{Proj}(H) \to \operatorname{Gr}(H)$, so the second map is also continuous. Therefore, φ is continuous. Obviously, φ is bijective.

The inverse map $\operatorname{Gr}^{(2)}(H) \to \operatorname{Proj}(H)$ is given by formula (9.2) and therefore is continuous. Thus the map $\operatorname{Proj}(H) \to \operatorname{Gr}^{(2)}(H)$ is a homeomorphism.

To prove that $\operatorname{Gr}^{(2)}(H)$ is open in $\operatorname{Gr}^2(H)$, take arbitrary $(L, M) \in \operatorname{Gr}^{(2)}(H)$. The operator $P_L - P_M$ is invertible by Corollary 9.2. Choose $\varepsilon > 0$ such that 2ε-neighbourhood of $P_L - P_M$ in $\mathcal{B}(H)$ consists of invertible operators. Then for any $L', M' \in \operatorname{Gr}(H)$ such that $\|P_L - P_{L'}\| < \varepsilon$, $\|P_M - P_{M'}\| < \varepsilon$ we have

$$\|(P_L - P_M) - (P_{L'} - P_{M'})\| \leqslant \|P_L - P_{L'}\| + \|P_M - P_{M'}\| < 2\varepsilon,$$

so $P_{L'} - P_{M'}$ is invertible. Applying again Corollary 9.2, we obtain $(L', M') \in \operatorname{Gr}^{(2)}(H)$. This completes the proof of the proposition for Hilbert spaces.

2. *Let now* H *be an arbitrary Banach space.* The continuity of the map $\operatorname{Im}\colon \operatorname{Proj}(H) \to \operatorname{Gr}(H)$ follows from the inequality $\hat{\delta}(\operatorname{Im} P, \operatorname{Im} Q) \leqslant \|P - Q\|$. As above, this implies that φ is a continuous bijection. The continuity of the map $\operatorname{Gr}^{(2)}(H) \to \operatorname{Proj}(H)$, $(L, M) \mapsto P_{L,M}$ follows from [Ne, Lemma 0.2]. By [GM, Lemma 1 and Theorem 2], $\operatorname{Gr}^{(2)}(H)$ is open in $\operatorname{Gr}^2(H)$. This completes the proof of the proposition for Banach spaces. □

Proposition 10.2. *Let* H *be a Banach space. Then the gap topology on* $\operatorname{Gr}(H)$ *coincides with the quotient topology induced by the map* $\operatorname{Im}\colon \operatorname{Proj}(H) \to \operatorname{Gr}(H)$, $P \mapsto \operatorname{Im} P$.

Proof. The projection $p_1\colon \operatorname{Gr}^2(H) \to \operatorname{Gr}(H)$ onto the first factor is an open continuous map. By Proposition 10.1, $\operatorname{Gr}^{(2)}(H)$ is open in $\operatorname{Gr}^2(H)$, so the restriction of p_1 to $\operatorname{Gr}^{(2)}(H)$ is also an open map. This restriction maps $\operatorname{Gr}^{(2)}(H)$ continuously and surjectively onto $\operatorname{Gr}(H)$. Therefore, the gap topology on $\operatorname{Gr}(H)$ coincides with the quotient topology induced by the map $p_1\colon \operatorname{Gr}^{(2)}(H) \to \operatorname{Gr}(H)$. To complete the proof, it is sufficient to apply the homeomorphism $\varphi\colon \operatorname{Proj}(H) \to \operatorname{Gr}^{(2)}(H)$ from Proposition 10.1. □

11 Injective maps of Banach spaces

Proposition 11.1. *Let* $j \in \mathcal{B}(H, H')$ *be an injective map of Banach spaces. Denote by* $\operatorname{Gr}_j(H)$ *the subspace of* $\operatorname{Gr}(H)$ *consisting of* L *with* $j(L) \in \operatorname{Gr}(H')$. *Then* $\operatorname{Gr}_j(H)$ *is open in* $\operatorname{Gr}(H)$ *and the natural inclusion* $j_*\colon \operatorname{Gr}_j(H) \hookrightarrow \operatorname{Gr}(H')$, $L \mapsto j(L)$, *is continuous.*

Proof. By Proposition 10.1, $\mathrm{Gr}^M(H)$ is open in $\mathrm{Gr}(H)$. Thus the statement of the proposition results from the following lemma.

Lemma 11.2. *Let* $L \in \mathrm{Gr}_j(H)$, *let* $M' \in \mathrm{Gr}(H')$ *be a complementary subspace to* $L' = j(L)$, *and let* $M = j^{-1}(M')$. *Then* $L \in \mathrm{Gr}^M(H) \subset \mathrm{Gr}_j(H)$, *and the restriction of* j_* *to* $\mathrm{Gr}^M(H)$ *is continuous.*

Proof of the Lemma. Denote by Q' the projection of H' onto L' along M'. By the Closed Graph Theorem, the bounded linear operator $j|_L : L \to L'$ is an isomorphism. Thus the composition $Q = (j|_L)^{-1} Q' j$ is a bounded operator on H. Obviously, Q is an idempotent, $\operatorname{Im} Q = L$, and $\ker Q = M$. This implies that (L, M) is a complementary pair of subspaces of H.

Let $N \in \mathrm{Gr}^M(H)$, $N' = j(N)$. Then $Q_N = jP_{N,M}(j|_L)^{-1}Q'$ is a bounded operator acting on H'. The kernel of Q_N is M' and the restriction of $Q_N^2 - Q_N$ to L' vanishes, so $Q_N^2 = Q_N$ and $Q_N \in \mathrm{Proj}(H')$. The image of Q_N contains in N' and $N' \cap M' = j(N \cap M) = 0$. Therefore, $Q_N = P_{N',M'}$, $N' = \operatorname{Im} Q_N \in \mathrm{Gr}(H')$, and $N \in \mathrm{Gr}_j(H)$.

By Proposition 10.1, the map $N \mapsto P_{N,M}$ is continuous. Thus the map $\mathrm{Gr}^M(H) \to \mathrm{Proj}(H')$, $N \mapsto Q_N$ is also continuous. Composing it with the continuous map $\operatorname{Im}\colon \mathrm{Proj}(H') \to \mathrm{Gr}(H')$, we obtain the continuity of the map $j_*\colon \mathrm{Gr}^M(H) \to \mathrm{Gr}(H')$, $N \mapsto j(N) = \operatorname{Im} Q_N$. This completes the proof of the lemma and of Proposition 11.1.
□

12 Closed operators

Let H and H' be Hilbert spaces. The space $\mathcal{C}(H, H')$ of closed linear operators from H to H' is the subspace of $\mathrm{Gr}(H \oplus H')$ consisting of closed subspaces of $H \oplus H'$ which injectively projects on H. An element of $\mathcal{C}(H, H')$ can be identified with a linear (not necessarily bounded) operator A acting to H' from (not necessarily closed or dense) subspace $\mathrm{dom}(A)$ of H such that the graph of A is a closed subspace of $H \oplus H'$.

All results of this section are valid for Banach spaces as well. *However, in this case the space* $\mathcal{C}(H, H')$ *as we define it (namely, as a the subspace of* $\mathrm{Gr}(H \oplus H')$*) does not contain all closed linear operators from* H *to* H', *but only those whose graphs are complemented subspaces of* $H \oplus H'$. Nevertheless, families of such operators often arise in applications, so these results can be used for them as well. For example, for Banach spaces H, H' and a linear operator A acting from $\mathcal{D} \subset H$ to H', if $\operatorname{Ker} A \subset H$ and $\operatorname{Im} A \subset H'$ are closed complemented subspaces, then the graph of A is a closed complemented subspace of $H \oplus H'$. In particular, every (not necessarily bounded) Fredholm operator has this property.

Proposition 12.1. *Let* H, H' *be Banach spaces. Then the map* $\mathcal{B}(H, H') \times \mathrm{Gr}(H) \to \mathcal{C}(H, H')$ *taking* $(A, \mathcal{D})$ *to* $A|_{\mathcal{D}}$ *is continuous.*

Proof. For each $A \in \mathcal{B}(H, H')$ we define the automorphism J_A of $H \oplus H'$ by the formula $J_A(u \oplus u') = u \oplus (u' - Au)$. Both $A \mapsto J_A$ and $A \mapsto J_A^{-1}$ are continuous maps from $\mathcal{B}(H, H')$ to $\mathcal{B}(H \oplus H')$. The formula $f(A, Q) = J_A^{-1} Q P_{H,H'} J_A$ defines the continuous map $f\colon \mathcal{B}(H, H') \times \mathrm{Proj}(H) \to \mathrm{Proj}(H \oplus H')$ (here $P_{H,H'}$ denotes the projection of $H \oplus H'$ on H along H'). Since J_A takes the graph of $A|_{\mathcal{D}}$ to $\mathcal{D} \oplus 0$ for each $\mathcal{D} \in \mathrm{Gr}(H)$, $f(A, Q)$ is the projection of $H \oplus H'$ onto the graph of $A|_{\mathrm{Im}\, Q}$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{B}(H, H') \times \mathrm{Proj}(H) & \xrightarrow{f} & \mathrm{Proj}(H \oplus H') \\ \big\downarrow{\scriptstyle \mathrm{Id} \times \mathrm{Im}} & & \big\downarrow{\scriptstyle \mathrm{Im}} \\ \mathcal{B}(H, H') \times \mathrm{Gr}(H) & \xrightarrow{g} & \mathrm{Gr}(H \oplus H') \end{array}$$

where g is the map taking the pair $(A, \mathcal{D})$ to the graph of $A|_{\mathcal{D}}$. The top and the right arrows of the diagram are continuous maps, while the left arrow is a quotient map by Proposition 10.2. Therefore, g is also continuous. This completes the proof of the proposition. □

Proposition 12.2. *Let W, H, H' be Banach spaces, and let $j \in \mathcal{B}(W, H)$ be injective. Denote by $\mathcal{C}_j(W, H')$ the subspace of $\mathcal{C}(W, H')$ consisting of operators $A\colon \mathrm{dom}(A) \to H'$ such that the operator $j_* A\colon j(\mathrm{dom}(A)) \to H'$, $j_* A = A \cdot j^{-1}$ lies in $\mathcal{C}(H, H')$. Then the natural inclusion $j_*\colon \mathcal{C}_j(W, H') \hookrightarrow \mathcal{C}(H, H')$ is continuous.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}(W, H') & \hookleftarrow & \mathcal{C}_j(W, H') & \xrightarrow{j_*} & \mathcal{C}(H, H') \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{Gr}(W \oplus H') & \hookleftarrow & \mathrm{Gr}_j(W \oplus H') & \xrightarrow{j_*} & \mathrm{Gr}(H \oplus H') \end{array}$$

The spaces above are just subspaces on the spaces below, and $\mathcal{C}_j(W, H') = \mathcal{C}(W, H') \cap \mathrm{Gr}_j(W \oplus H')$. By Proposition 11.1, the map $j_*\colon \mathrm{Gr}_j(W \oplus H') \to \mathrm{Gr}(H \oplus H')$ is continuous. So the restriction of j_* to $\mathcal{C}_j(W, H') \subset \mathrm{Gr}_j(W \oplus H')$ is also continuous. This completes the proof of the proposition. □

13 Differential and pseudo-differential operators

The results of the previous section can be used for differential and pseudo-differential operators acting between sections of vector bundles over M. To achieve continuity of the corresponding families of closed operators, the relevant topology on the space of differential operators will be the C^0_b-topology on their coefficients.

General framework. Let M be a smooth Riemannian manifold and E, E' be smooth Hermitian vector bundles over M. For an integer $d \geqslant 1$, we denote by $\mathrm{Op}^d(E, E')$ the set of all pairs $(A, \mathcal{D})$ such that

- A is a bounded operator from $H^d(E)$ to $L^2(E')$,

- $\mathcal{D}$ is a closed subspace of $H^d(E)$, and
- the restriction $A|_{\mathcal{D}}$ of A to the domain $\mathcal{D}$ is closed as an operator from $L^2(E)$ to $L^2(E')$.

We equip $\mathrm{Op}^d(E,E')$ with the topology induced by the inclusion

$$\mathrm{Op}^d(E,E') \hookrightarrow \mathcal{B}(H^d(E), L^2(E')) \times \mathrm{Gr}(H^d(E)).$$

Here $L^2(E)$ is the Hilbert space of L^2-sections of E and $H^d(E)$ is the d-th order Sobolev space of sections of E.

Proposition 13.1. *The map* $\mathrm{Op}^d(E,E') \to \mathcal{C}(L^2(E), L^2(E'))$ *taking* $(A,\mathcal{D})$ *to* $A|_{\mathcal{D}}$ *is continuous.*

Proof. Take $W = H^d(E)$, $H = L^2(E)$, $H' = L^2(E')$, and let j be the natural embedding $W \hookrightarrow H$. By Proposition 12.1, the map $\mathrm{Op}^d(E,E') \subset \mathcal{B}(W,H') \times \mathrm{Gr}(W) \to \mathcal{C}(W,H')$ is continuous. By definition of $\mathrm{Op}^d(E,E')$, the image of this map is contained in $\mathcal{C}_j(W,H')$. By Proposition 12.2, the map $j_*\colon \mathcal{C}_j(W,H') \to \mathcal{C}(H,H')$ is continuous. Combining all this, we obtain the continuity of the map $\mathrm{Op}^d(E,E') \to \mathcal{C}(H,H')$. □

This general result can be applied to differential or pseudo-differential operators A of order d with domains $\mathcal{D}$ given by boundary conditions. We show below how Proposition 13.1 can be applied to boundary value problems for first order differential operators, in particular local boundary value problems. We omit a discussion of higher order operators because boundary conditions are slightly more complicated in that case; however, Proposition 13.1 works for higher order operators as well.

Boundary value problems for first order operators. Suppose now that M is a compact manifold. Denote by E_∂ the restriction of E to the boundary ∂M.

Let $A \in \mathcal{B}(H^1(E), L^2(E'))$. In particular, A can be a first order differential operator with continuous coefficients. For a closed subspace $\mathcal{L}$ of $H^{1/2}(E_\partial)$ we denote by $A_{\mathcal{L}}$ the operator A with the domain

$$\mathrm{dom}\,(A_{\mathcal{L}}) = \{u \in H^1(E) \colon \tau(u) \in \mathcal{L}\},$$

where $\tau\colon H^1(E) \to H^{1/2}(E_\partial)$ is the trace map extending by continuity the restriction map $C^\infty(E) \to C^\infty(E_\partial)$, $u \mapsto u|_{\partial M}$.

Let $\widetilde{\mathrm{Op}}(E,E')$ denotes the subspace of $\mathcal{B}(H^1(E), L^2(E')) \times \mathrm{Gr}(H^{1/2}(E_\partial))$ consisting of pairs $(A,\mathcal{L})$ such that the operator $A_{\mathcal{L}}$ is closed.

Proposition 13.2. *The map*

$$\widetilde{\mathrm{Op}}(E,E') \to \mathcal{C}(L^2(E), L^2(E')), \quad (A,\mathcal{L}) \mapsto A_{\mathcal{L}}$$

is continuous.

Proof. The inverse image $\tau^{-1}(\mathcal{L})$ is a closed subspace of $H^1(E)$. Since τ is bounded and surjective, the map

$$\tau^*\colon\ \mathrm{Gr}(H^{1/2}(E_\partial)) \to \mathrm{Gr}(H^1(E)), \quad \mathcal{L} \mapsto \tau^{-1}(\mathcal{L}),$$

is continuous. Hence the map $\widetilde{\mathrm{Op}}(E,E') \to \mathrm{Op}^1(E,E')$ taking $(A,\mathcal{L})$ to $(A,\tau^{-1}(\mathcal{L}))$ is also continuous. It remains to apply Proposition 13.1. □

Local boundary value problems for first order operators. Denote by $\mathrm{Ell}(E,E')$ the set of all first order elliptic differential operators with smooth coefficients acting from sections of E to sections of E'.

Let L be a smooth subbundle of E_∂. The Sobolev space $H^{1/2}(L)$ can be naturally identified with the closed subspace of $H^{1/2}(E_\partial)$ via the map $H^{1/2}(L) \ni u \mapsto u \oplus 0 \in H^{1/2}(L) \oplus H^{1/2}(L^\perp) = H^{1/2}(E_\partial)$. This allows to associate with a pair (A,L) the unbounded operator A_L acting as A on the domain

$$\mathrm{dom}\,(A_L) = \left\{ u \in H^1(E) \colon \tau(u) \in H^{1/2}(L) \right\}.$$

Denote by $\widetilde{\mathrm{Ell}}(E,E')$ the set of all pairs (A,L) with $A \in \mathrm{Ell}(E,E')$ and L a smooth subbundle of E_∂ such that the unbounded operator A_L is closed.

Remark. By the classical theory of elliptic operators, A_L is closed for L satisfying the Shapiro-Lopatinskii condition. See, for example, Proposition 14.2 below, where it is proven for self-adjoint operators. Closedness of a non-self-adjoint A_L can be proven along the same lines, or can be obtained directly from Proposition 14.2 by replacing a pair (A,L) with the pair $(A',L') \in \widetilde{\mathrm{Ell}}(E \oplus E')$, where $A' = \left(\begin{smallmatrix} 0 & A^t \\ A & 0 \end{smallmatrix}\right)$ and $L' = L \oplus (\sigma_A(n)L)^\perp \subset E_\partial \oplus E'_\partial$.

Equip $\widetilde{\mathrm{Ell}}(E,E')$ with the C^0-topology on coefficients of operators and the C^1-topology on boundary conditions, that is, the topology induced by the inclusion

$$\widetilde{\mathrm{Ell}}(E,E') \hookrightarrow \mathcal{B}(H^1(E), L^2(E')) \times C^1(\mathrm{Gr}(E_\partial)).$$

Here $\mathrm{Gr}(E_\partial)$ denotes the smooth vector bundle over ∂M whose fiber over $x \in \partial M$ is the Grassmanian $\mathrm{Gr}(E_x)$, and sections of $\mathrm{Gr}(E_\partial)$ are identified with subbundles of E_∂.

Proposition 13.3. *The natural inclusion* $\widetilde{\mathrm{Ell}}(E,E') \hookrightarrow \mathcal{C}(L^2(E), L^2(E'))$, $(A,L) \mapsto A_L$ *is continuous.*

Proof. It is an immediate corollary of the following lemma applied to $N = \partial M$ and $F = E_\partial$ and of Proposition 13.2.

Lemma 13.4. *Let* F *be a smooth Hermitian vector bundle over a smooth closed Riemannian manifold* N. *Then the map*

$$C^{\infty,1}(\mathrm{Gr}(F)) \to \mathrm{Gr}(H^{1/2}(F)), \tag{13.1}$$

taking a smooth subbundle L *of* F *to* $H^{1/2}(L) \subset H^{1/2}(F)$, *is continuous. Here* $C^{\infty,1}(\mathrm{Gr}(F))$ *denotes the space of smooth sections of* $\mathrm{Gr}(F)$ *with the* C^1*-topology, that is, the topology induced by the embedding* $C^\infty(\mathrm{Gr}(F)) \hookrightarrow C^1(\mathrm{Gr}(F))$.

Proof. The operator of multiplication by a C^1-function $N \to \mathbb{C}$ is a bounded operator on $H^s(N)$ for every $s \in [0,1]$. In particular, it is bounded as an operator acting on $H^{1/2}(N)$, and the corresponding inclusion $C^1(N) \hookrightarrow \mathcal{B}\left(H^{1/2}(N)\right)$ is continuous. Passing to bundles, we obtain the natural continuous inclusion

$$C^1(\mathrm{End}(F)) \hookrightarrow \mathcal{B}\left(H^{1/2}(F)\right). \tag{13.2}$$

The smooth map $P\colon \mathrm{Gr}(\mathbb{C}^n) \to \mathrm{End}(\mathbb{C}^n)$, $V \mapsto P_V$, induces the continuous map

$$P_*\colon C^1(\mathrm{Gr}(F)) \hookrightarrow C^1(\mathrm{End}(F)),$$

which carries a subbundle L of F to the orthogonal projection P_*L of F onto L. Composing it with the continuous inclusion (13.2), we obtain the continuous map

$$Q\colon C^1(\mathrm{Gr}(F)) \hookrightarrow \mathcal{B}\left(H^{1/2}(F)\right).$$

For each smooth subbundle L of F the bounded operator $Q(L)$ is an idempotent with the image $H^{1/2}(L)$. By Proposition 10.1(1), the map

$$\mathrm{Im}\colon \mathrm{Proj}\left(H^{1/2}(F)\right) \to \mathrm{Gr}\left(H^{1/2}(F)\right),$$

is continuous. Composing it with Q, we obtain continuity of (13.1). This completes the proof of the lemma. □

Part IV

Elliptic local boundary value problems

14 Local boundary value problems

Let M be a smooth compact connected oriented manifold with non-empty boundary ∂M and a fixed Riemannian metric, and let E be a smooth Hermitian complex vector bundle over M. We denote by E_∂ the restriction of E to the boundary ∂M.

Operators. Let A be a first order elliptic differential operator acting on sections of E. Recall that an operator A is called elliptic if its (principal) symbol $\sigma_A(\xi)$ is non-degenerate for every non-zero cotangent vector $\xi \in T^*M$. Throughout the main part of the thesis (except for Part III) all differential operators are supposed to have smooth (C^∞) coefficients.

An operator A is called formally self-adjoint if it is symmetric on the domain $C_0^\infty(E)$, that is, if $\int_M \langle Au, v\rangle \,\mathrm{dvol} = \int_M \langle u, Av\rangle \,\mathrm{dvol}$ for any smooth sections u, v of E with compact supports in $M \setminus \partial M$.

Local boundary conditions. The differential operator A with the domain $C_0^\infty(E)$ is an unbounded operator on the Hilbert space $L^2(E)$ of L^2-sections of E. This operator can be extended to a closed operator on $L^2(E)$ in various ways, by imposing appropriate boundary conditions. We will consider only local boundary conditions that are defined by smooth subbundles of E_∂. For such a subbundle L, the corresponding unbounded operator A_L on $L^2(E)$ has the domain

$$\operatorname{dom}(A_L) = \{u \in H^1(E)\colon\ u|_{\partial M} \text{ is a section of } L\}, \tag{14.1}$$

where $H^1(E)$ denotes the first order Sobolev space (the space of sections of E which are in L^2 together with all their first derivatives). We will often identify a pair (A, L) with the operator A_L.

To give a precise meaning to the notation in the right-hand side of (14.1), recall that the restriction map $C^\infty(E) \to C^\infty(E_\partial)$ taking a section u to $u|_{\partial M}$ extends continuously to the trace map $\tau\colon H^1(E) \to H^{1/2}(E_\partial)$. The smooth embedding $L \hookrightarrow E_\partial$ defines the natural inclusion $H^{1/2}(L) \hookrightarrow H^{1/2}(E_\partial)$. By the condition "$u|_{\partial M}$ is a section of L" in (14.1) we mean that the trace $\tau(u)$ lies in the image of this inclusion.

Decomposition of E. We will use the following properties of elliptic symbols.

Proposition 14.1. *Let $\sigma \in \mathrm{Hom}(T^*M, \mathrm{End}(E))$ be a symbol of first order elliptic operator. Let Π be an oriented two-dimensional plane in the cotangent bundle T_x^*M, $x \in M$. Then for any positive oriented frame (e_1, e_2) in Π the operator*

$$Q = \sigma(e_1)^{-1}\sigma(e_2) \in \mathrm{End}(E_x)$$

has no eigenvalues on the real axis. It defines the direct sum decomposition $E_x = E^+ \oplus E^-$ (not necessarily orthogonal), where E^+ and E^- are spanned by the generalized eigenspaces of Q corresponding to the eigenvalues with positive and negative imaginary part respectively. This decomposition depends only on Π and is independent of the choice of a frame (e_1, e_2).

If additionally σ is self-adjoint, then the ranks of E^+ and E^- are equal (so the rank of E is even), and for every non-zero $\xi \in \Pi$ the symbol $\sigma(\xi)$ takes E^+ and E^- to their orthogonal complements in E_x.

Proof. 1. Since σ is elliptic, the operator $Q - t = \sigma(e_1)^{-1}\sigma(e_2 - te_1)$ is invertible for any $t \in \mathbb{R}$. Hence Q has no eigenvalues on the real axis and $E_x = E^+ \oplus E^-$.

If we change (e_1, e_2) to $(e_1, e_2 + te_1)$, $t \in \mathbb{R}$, then Q is changed to $Q + t\,\mathrm{Id}$. If we change (e_1, e_2) to $(e_1 + te_2, e_2)$ then Q is changed to $(Q^{-1} + t\,\mathrm{Id})^{-1}$. In both cases E^+ and E^- do not change. Therefore, they do not change at any change of the frame (e_1, e_2) preserving orientation, and thus depend only on Π.

2. Suppose now that σ is self-adjoint, that is, $\sigma(\xi)$ is self-adjoint for every $\xi \in T^*M$. Let $\xi \in \Pi$ be a non-zero vector. Choose a positive oriented frame (e_1, e_2) in Π such that $e_1 = \xi$. Denote $\sigma_i = \sigma(e_i)$, $V_{\lambda,k} = \mathrm{Ker}(Q - \lambda)^k$, and $V_\lambda = V_{\lambda, \dim E}$. We prove by induction that $\sigma_1 V_\lambda$ is orthogonal to V_μ for any $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \overline{\mu}$. Indeed, $\sigma_1 V_{\lambda,0} = 0$ is orthogonal to $V_{\mu,0} = 0$. Suppose that $\sigma_1 V_{\lambda,l}$ is orthogonal to $V_{\mu,m}$ for all $l, m \geqslant 0$, $l + m < k$. Then for $l + m = k$, $u \in V_{\lambda,l}$, $v \in V_{\mu,m}$ we have

$$(\lambda - \overline{\mu})\,\langle \sigma_1 u, v\rangle = \langle \sigma_1 \lambda u, v\rangle - \langle \sigma_1 u, \mu v\rangle + \langle u, \sigma_2 v\rangle - \langle \sigma_2 u, v\rangle =$$
$$= \langle \sigma_1 \lambda u, v\rangle - \langle \sigma_1 u, \mu v\rangle + \langle \sigma_1 u, Qv\rangle - \langle \sigma_1 Q u, v\rangle = \langle \sigma_1 u, (Q - \mu)v\rangle - \langle \sigma_1 (Q - \lambda)u, v\rangle = 0$$

by induction assumption, since $(Q - \mu)v \in V_{\mu, m-1}$ and $(Q - \lambda)u \in V_{\lambda, l-1}$. Thus $\sigma_1 V_\lambda$ is orthogonal to V_μ if $\lambda \neq \overline{\mu}$.

The subspace E^+ is spanned by $\bigcup V_\lambda$ with λ running over all the eigenvalues of Q with positive imaginary parts. For every pair λ, μ of such eigenvalues (not necessarily distinct) we have $\lambda \neq \overline{\mu}$, so $\sigma_1 E^+$ is orthogonal to E^+. Similarly, $\sigma_1 E^-$ is orthogonal to E^-. We have

$$2 \dim E^+ = \dim E^+ + \dim(\sigma_1 E^+) \leqslant \dim E^+ + \dim (E^+)^\perp = \dim E_x$$

and, similarly, $2 \dim E^- \leqslant \dim E_x$. On the other hand, $\dim E^+ + \dim E^- = \dim E_x$. Therefore, $\dim E^+ = \dim E^- = \dim E_x / 2$. □

Elliptic boundary conditions Let A be a first order elliptic operator acting on sections of E. The inverse image of a non-zero cotangent vector $\xi \in T^*_x \partial M$ under the restriction map $T^*_x M \to T^*_x \partial M$ is an affine line in $T^*_x M$ parallel to the outward conormal n_x. Denote by Π_ξ the two-dimensional vector subspace of $T^*_x M$ spanned by this line. Applying Proposition 14.1 to the plane Π_ξ, we obtain the decomposition of E_x into the direct sum $E^+(\xi) \oplus E^-(\xi)$.

A local boundary condition L is called elliptic for A if

(14.2) $L_x \cap E^+(\xi) = 0$ and $L_x + E^+(\xi) = E_x$ for every non-zero $\xi \in T^*_x \partial M, x \in \partial M$.

If L is elliptic for A, then the adjoint to A_L is A^t_N, where A^t is the differential operator formally adjoint to A and $N = (\sigma(n)L)^\perp$.

Self-adjoint elliptic boundary conditions Suppose now that an elliptic operator A is formally self-adjoint. Then the conormal symbol $\sigma(n)$ of A defines a symplectic structure on fibers of E_∂ given by the symplectic 2-form $\omega_x(u,v) = \langle i\sigma(n)u, v\rangle$ for $u, v \in E_x$, $x \in \partial M$, where n is the outward conormal to ∂M. By Proposition 14.1 both $E^+(\xi)$ and $E^-(\xi)$ are Lagrangian subspaces with respect to this symplectic structure.

The differential operator A with the domain $C_0^\infty(E)$ is a symmetric unbounded operator on the Hilbert space $L^2(E)$ of L^2-sections of E. This operator can be extended to a regular self-adjoint operator on $L^2(E)$ by imposing appropriate boundary conditions. For L satisfying ellipticity condition (14.2), the operator A_L is self-adjoint if and only if L is a Lagrangian subbundle of E_∂.

For a Lagrangian subbundle L condition (14.2) can be written in simpler form:

$$(14.3) \qquad L_x \cap E^+(\xi) = 0 \ \text{ for every non-zero } \xi \in T_x^*\partial M, x \in \partial M.$$

Indeed, rank of both L_x and $E^+(\xi)$ is half of rank E_x. Therefore, $L_x \cap E^+(\xi) = 0$ if and only if $L_x + E^+(\xi) = E_x$.

Finally, we obtain the following description of self-adjoint elliptic local boundary value problems.

Proposition 14.2. *Let A be a first order formally self-adjoint elliptic differential operator acting on sections of E. Let L be a smooth Lagrangian subbundle of E_∂ satisfying condition* (14.3)*. Then A_L is a regular Fredholm self-adjoint operator on $L^2(E)$. Moreover, A_L has compact resolvents, that is, $(A_L + i)^{-1}$ is a compact operator on $L^2(E)$.*

Proof. Denote by $\mathcal{D}$ the domain of A_L given by formula (14.1). It is dense in $L^2(E)$ and closed in $H^1(E)$. Equip $\mathcal{D}$ with the topology induced from $H^1(E)$.

Let $\tau\colon H^1(E) \to H^{1/2}(E_\partial)$ be the trace map, and let P be the bundle endomorphism projecting E_∂ on $L^\perp$ along L. Condition (14.2) means that $P\colon E^+(\xi) \to L_x^\perp$ is bijective for every non-zero $\xi \in T_x^*\partial M$. It follows by [Hö, Theorem 20.1.2] that the operator $A \oplus P\tau\colon H^1(E) \to L^2(E) \oplus H^{1/2}(L^\perp)$ is Fredholm. Its restriction to the kernel of $P\tau$ is also Fredholm. But this restriction coincides with A_L considered as a bounded operator from $\mathcal{D}$ to $L^2(E)$. Hence A_L is Fredholm.

In particular, $V = \operatorname{Im}(A_L)$ is a closed subspace of $L^2(E)$. Let U be the orthogonal complement of the kernel of A_L in $L^2(E)$. The restriction $\bar{A}$ of A to U is injective with the image V. Therefore, the inverse operator $\bar{A}^{-1}\colon V \to U$ is bounded and its graph is closed in $V \times U$. Equivalently, the graph of $\bar{A}$ is closed in $U \times V$, which is a closed subspace of $L^2(E)^2$. The graph of A_L is the orthogonal sum of $\ker(A_L) \times \{0\}$ with the graph of $\bar{A}$ and therefore is closed in $L^2(E)^2$. In other words, the operator A_L is closed.

Green's formula implies that A_L is symmetric. Let $(u, v) \in L^2(E)^2$ be an arbitrary point of the graph of the adjoint operator. This means that for each $w \in \operatorname{dom}(A_L)$ we have $\langle u, Aw\rangle = \langle v, w\rangle$. By [LP, Theorem 1], (u, v) lies in the closure of the graph of A_L. (The statement of this theorem of Lax and Phillips concerns only smooth domains in Euclidean spaces and trivial vector bundles. But its proof is local, so it works for the general case without change.) Since A_L is closed, $u \in \operatorname{dom}(A_L)$. Therefore, A_L is self-adjoint.

The operator A_L is bounded as an operator from the Hilbert space $\mathcal{D}$ to $L^2(E)$. Since A_L is a closed self-adjoint operator on $L^2(E)$, the bounded operator $A_L + i\colon \mathcal{D} \to L^2(E)$ is bijective and the inverse $(A_L + i)^{-1}$ is a bounded operator from $L^2(E)$ to $\mathcal{D}$ [Kat, Theorem V.3.16]. Composing it with the compact embedding $\mathcal{D} \subset H^1(E) \hookrightarrow L^2(E)$, we see that $(A_L + i)^{-1}$ is compact as an operator on $L^2(E)$. This completes the proof of the proposition. □

The space of boundary value problems Denote by $\overline{\mathrm{Ell}}(E)$ the set of all pairs (A, L) satisfying conditions of Proposition 14.2. The following result is a particular case of Proposition 13.3 from Part III.

Proposition 14.3. *For the set $\overline{\mathrm{Ell}}(E)$ equipped with the C^0-topology on coefficients of operators and the C^1-topology on boundary conditions, the natural inclusion $\overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_F(L^2(E))$, $(A, L) \mapsto A_L$ is continuous.*

Equivalently, the C^0-topology on coefficients of operators can be described as the topology induced by the inclusion $\mathrm{Ell}(E) \hookrightarrow \mathcal{B}(H^1(E), L^2(E))$.

15 Boundary value problems on a surface

From now on we will consider only the case of ***dimension two***, that is, M will be a smooth compact connected oriented ***surface*** with non-empty boundary ∂M and a fixed Riemannian metric.

Let E be a smooth Hermitian complex vector bundle over M. Denote by $\mathrm{Ell}(E)$ the set of first order formally self-adjoint elliptic differential operators with smooth coefficients acting on sections of E.

Decomposition of a bundle. Since M is now two-dimensional, Proposition 14.1 allows to define the *global* decomposition of E.

Proposition 15.1. *Let $A \in \mathrm{Ell}(E)$. Then the symbol σ of A defines the decomposition of E into the direct sum (not necessarily orthogonal) of two smooth subbundles $E^+ = E^+(\sigma)$ and $E^- = E^-(\sigma)$ such that the following conditions hold:*

1. *E^+_x and E^-_x are spanned by the generalized eigenspaces of $Q_x = \sigma(e_1)^{-1}\sigma(e_2)$ as in Proposition 14.1, where (e_1, e_2) is an arbitrary positive oriented frame in T^*_xM.*
2. *Ranks of E^+ and E^- are equal, so the rank of E is even.*
3. *For every non-zero $\xi \in T^*_xM$ the symbol $\sigma(\xi)$ takes E^+_x and E^-_x to their orthogonal complements in E_x.*

Proof. The main part of the statement follows from Proposition 14.1. It remains to show that E^+_x and E^-_x are fibers of smooth vector bundles E^+ and E^-. Choosing a local smooth frame (e_1, e_2) in T^*M, we see that E^+_x and E^-_x smoothly depend on Q_x, which in turn smoothly depends on x. □

Self-adjoint elliptic boundary conditions. Denote by E_∂^+, respectively E_∂^- the restriction of E^+, respectively E^- to ∂M. As before, the conormal symbol $\sigma(n)$ defines the symplectic structure on the fibers of E_∂, and E_∂^+, E_∂^- are transversal Lagrangian subbundles of E_∂.

The orientation on M induces the orientation on ∂M. Fibers of $E_\partial^\pm$ can be written as $E_x^+ = E^+(\xi)$ and $E_x^- = E^-(\xi)$, where ξ is a positive vector in the oriented one-dimensional space $T_x^*\partial M$. The identity $E^+(\xi) = E^-(-\xi)$ allows to write ellipticity condition (14.2) in a simpler form:

$$L \cap E_\partial^+ = L \cap E_\partial^- = 0 \;\text{ and }\; L + E_\partial^+ = L + E_\partial^- = E_\partial. \tag{15.1}$$

If L is Lagrangian, then condition (15.1) can be simplified even further, cf. (14.3):

$$L \cap E_\partial^+ = L \cap E_\partial^- = 0.$$

As before, we denote by $\overline{\mathrm{Ell}}(E)$ the set of all pairs (A, L) such that $A \in \mathrm{Ell}(E)$ and L is a smooth Lagrangian subbundle of E_∂ satisfying condition (15.1). Proposition 14.2 then takes the following form.

Proposition 15.2. *For every* $(A, L) \in \overline{\mathrm{Ell}}(E)$ *the unbounded operator* A_L *is a regular self-adjoint operator on* $L^2(E)$ *with compact resolvents.*

The correspondence between boundary conditions and automorphisms of E_∂^-. For every elliptic (not necessarily self-adjoint) symbol σ there is a one-to-one correspondence between subbundles L of E_∂ satisfying condition (15.1) and bundle isomorphisms $R\colon E_\partial^- \to E_\partial^+$. Namely, L is the graph of R in $E_\partial^- \oplus E_\partial^+ = E_\partial$. Equivalently, $-R$ is the projection of E_∂^- onto E_∂^+ along L.

If additionally σ is self-adjoint, then one can move further and construct a one-to-one correspondence between *Lagrangian* subbundles L of E_∂ satisfying (15.1) and *self-adjoint* bundle automorphisms T of E_∂^-.

Let us first describe this correspondence in the case of *mutually orthogonal* E_∂^+ and E_∂^- (this holds, in particular, for Dirac type operators). Composing R with $i\sigma(n)\colon E_\partial^+ \to (E_\partial^+)^\perp = E_\partial^-$, we obtain the bundle automorphism T of E_∂^-. Conversely, with every bundle automorphism T of E_∂^- we associate the subbundle L of E_∂ given by the formula

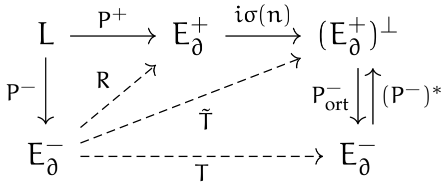
$$L = \left\{ u^+ \oplus u^- \in E_\partial^+ \oplus E_\partial^- = E_\partial \colon\; i\sigma(n)u^+ = Tu^- \right\}. \tag{15.2}$$

As Proposition 15.3 below shows, T is self-adjoint if and only if L is Lagrangian, so we obtain a bijection between the set of all self-adjoint elliptic local boundary conditions for A and the set of all self-adjoint bundle automorphisms of E_∂^-.

In the general case, where E_∂^+ and E_∂^- can be non-orthogonal, this construction should be slightly modified. The composition $\tilde{T} = i\sigma(n)R$ acts from E_∂^- to $(E_\partial^+)^\perp$, which now does not coincide with E_∂^-. In order to correct this, we compose $\tilde{T}$ with the orthogonal projection P_{ort}^- of E_∂ onto E_∂^-. Since $E_\partial = (E_\partial^-)^\perp \oplus (E_\partial^+)^\perp$, the restriction of P_{ort}^- to

$(E_\partial^+)^\perp$ is an isomorphism $(E_\partial^+)^\perp \to E_\partial^-$. Finally, we define the bundle automorphism $T = P_{\mathrm{ort}}^- \circ \tilde{T}$ of E_∂^-, so that the following diagram becomes commutative.

(15.3)



Proposition 15.3. *Let $A \in \mathrm{Ell}(E)$. Denote by P^+ the projection of E_∂ onto E_∂^+ along E_∂^- and by $P^- = 1 - P^+$ the projection of E_∂ onto E_∂^- along E_∂^+. Then the following hold.*

1. *There is a one-to-one correspondence between smooth subbundles L of E_∂ satisfying condition (15.1) and smooth bundle automorphisms T of E_∂^-. This correspondence is given by the formula*

   (15.4) $$L = \operatorname{Ker} P_T \quad \textit{with} \quad P_T = P^+\left(1 + i\sigma(n)^{-1} T P^-\right),$$

   where P_T is the projection of E_∂ onto E_∂^+ along L.

2. *For L and T as above, L is Lagrangian if and only if T is self-adjoint.*

If E_∂^+ and E_∂^- are mutually orthogonal, then (15.4) is equivalent to (15.2).

In the rest of the thesis we will sometimes write an element of $\overline{\mathrm{Ell}}(E)$ as (A, T) instead of (A, L).

Proof. The adjoint $(P^-)^*$ projects E_∂ onto $(E_\partial^+)^\perp$ along $(E_\partial^-)^\perp$, so its restriction to E_∂^- is the inverse of $P_{\mathrm{ort}}^-\colon (E_\partial^+)^\perp \to E_\partial^-$. All three solid arrows at the right half of Diagram (15.3) are smooth bundle isomorphisms.

By Proposition 15.1 the conormal symbol $\sigma(n)$ takes E_∂^- and E_∂^+ to their orthogonal complements. So $(P^-)^* = \sigma(n) P^+ \sigma(n)^{-1}$ and $\sigma(n)^{-1}(P^-)^* = P^+\sigma(n)^{-1}$. Therefore, (15.4) can be equivalently written as

(15.5) $$P_T = P^+ + i\sigma(n)^{-1}(P^-)^* T P^-.$$

1. Let L be a smooth subbundle of E_∂ satisfying (15.1). Then both solid arrows at the left half of Diagram (15.3) are smooth bundle isomorphisms. There is a smooth automorphism T of E_∂^- making this diagram commutative, and such an automorphism is unique. Substituting $R = (i\sigma(n))^{-1}(P^-)^* T$ to $L = \operatorname{Ker}(P^+ - RP^-)$, we obtain $L = \operatorname{Ker}(P^+ + i\sigma(n)^{-1}(P^-)^* T P^-) = \operatorname{Ker} P_T$.

Conversely, let T be a smooth automorphism of E_∂^-. The image of P_T is contained in E_∂^+, while the restriction of P_T to E_∂^+ is the identity. It follows that $P_T^2 = P_T$, that is, P_T is the projection of E_∂ onto E_∂^+ along $L = \operatorname{Ker} P_T$. This implies $L \cap E_\partial^+ = 0$ and $L + E_\partial^+ = E_\partial$. The projection P_T smoothly depends on $x \in \partial M$ and has constant rank, so L is a smooth subbundle of E_∂ with $\operatorname{rank} L = \operatorname{rank} E_\partial - \operatorname{rank} E_\partial^+ = \operatorname{rank} E_\partial^-$. If $u \in L \cap E_\partial^-$, then $P^+u = 0$ and $Tu = P_{\mathrm{ort}}^- i\sigma(n) P^+ u = 0$. Since T is invertible, $L \cap E_\partial^- = 0$. This completes the proof of clause 1.

2. Let L, T be as in clause 1 and $u_1, u_2 \in L$. For $u_j^- = P^- u_j$ and $u_j^+ = P^+ u_j$ we have

$$\left\langle T u_1^-, u_2^- \right\rangle = \left\langle \tilde{T} u_1^-, u_2^- \right\rangle = \left\langle i\sigma(n) u_1^+, u_2^- \right\rangle = \left\langle i\sigma(n) u_1^+, u_2 \right\rangle, \tag{15.6}$$

using the orthogonality of $\tilde{T}u_1^- - Tu_1^- = (1 - P^-_{\mathrm{ort}})\tilde{T}u_1^- \in (E_\partial^-)^\perp$ to $u_2^- \in E_\partial^-$ and the orthogonality of $i\sigma(n)u_1^+ \in (E_\partial^+)^\perp$ to $u_2 - u_2^- = u_2^+ \in E_\partial^+$. Similarly,

$$\left\langle u_1^-, T u_2^- \right\rangle = \left\langle u_1^-, i\sigma(n) u_2^+ \right\rangle = -\left\langle i\sigma(n) u_1^-, u_2^+ \right\rangle = -\left\langle i\sigma(n) u_1^-, u_2 \right\rangle. \tag{15.7}$$

Subtracting (15.7) from (15.6), we obtain

$$\langle i\sigma(n) u_1, u_2 \rangle = \left\langle TP^- u_1, P^- u_2 \right\rangle - \left\langle P^- u_1, TP^- u_2 \right\rangle \quad \text{for all } u_1, u_2 \in L. \tag{15.8}$$

If L is Lagrangian, then (15.8) implies self-adjointness of T, since $P^-: L \to E_\partial^-$ is surjective. Conversely, if T is self-adjoint, then (15.8) implies $i\sigma(n)L \subset L^\perp$; taking into account that $\operatorname{rank} L = \operatorname{rank} E_\partial/2$, we see that L is Lagrangian.

3. If E_∂^+ and E_∂^- are mutually orthogonal, then $(P^-)^*: E_\partial^- \to (E_\partial^+)^\perp$ is the identity, and (15.5) takes the form (15.2). This completes the proof of the proposition. □

The subbundle $F(A, L)$. With every $(A, L) \in \overline{\mathrm{Ell}}(E)$ we associate the smooth subbundle $F(A, L)$ of E_∂^- as follows. Let T be the self-adjoint automorphism of E_∂^- given by formula (15.4). We define F_x as the invariant subspace of T_x spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues. Subspaces F_x of E_x^- smoothly depend on $x \in \partial M$ and therefore are fibers of the smooth subbundle $F = F(A, L)$ of E_∂^-.

Being a subbundle of E_∂^-, $F(A, L)$ is also a smooth subbundle of E_∂. Sometimes it will be more convenient for us to consider $F(A, L)$ as a subbundle of E_∂.

16 The space of boundary value problems on a surface

Topology on $\overline{\mathrm{Ell}}(E)$. In section 14 we used the C^0-topology on coefficients of operators. We will compute the spectral flow for the paths in $\overline{\mathrm{Ell}}(E)$ which are continuous in a slightly stronger topology, namely the C^1-topology on symbols and the C^0-topology on free terms of operators. Let us describe this more precisely.

For a smooth complex vector bundle V over a smooth manifold N, we denote by $\mathrm{Gr}(V)$ the smooth bundle over N whose fiber over $x \in N$ is the complex Grassmanian $\mathrm{Gr}(V_x)$. In the same manner we define the smooth bundle $\mathrm{End}(V)$ of fiber endomorphisms. We identify sections of $\mathrm{Gr}(V)$ with subbundles of V and sections of $\mathrm{End}(V)$ with bundle endomorphisms of V.

Let $r = (r_1, r_0)$ be a couple of integers, $r_1 \geqslant r_0 \geqslant 0$. Denote by $\mathrm{Ell}^r(E)$ the set $\mathrm{Ell}(E)$ equipped with the C^{r_1}-topology on symbols and the C^{r_0}-topology on free terms of operators.

To be more precise, notice that the tangent bundle TM is trivial since M is a surface with non-empty boundary. Thus we can choose smooth global sections e_1, e_2 of TM such that $e_1(x)$, $e_2(x)$ are linear independent for any $x \in M$. Choose a smooth unitary connection ∇ on E. Each $A \in \mathrm{Ell}(E)$ can be written uniquely as $A = \sigma_1\nabla_1 + \sigma_2\nabla_2 + a$, where the symbol components $\sigma_i = \sigma_A(e_i)$ are self-adjoint bundle automorphisms of E, $\nabla_i = \nabla_{e_i}$, and the free term a is a bundle endomorphism. Therefore the choice of (e_1, e_2, ∇) defines the inclusion

$$\mathrm{Ell}(E) \hookrightarrow C^\infty(\mathrm{End}(E))^2 \times C^\infty(\mathrm{End}(E)), \quad \sigma_1\nabla_1 + \sigma_2\nabla_2 + a \mapsto (\sigma_1, \sigma_2, a),$$

where $C^\infty(\mathrm{End}(E))$ denotes the space of smooth sections of $\mathrm{End}(E)$. We equip $\mathrm{Ell}(E)$ with the topology induced by the inclusion

$$\mathrm{Ell}(E) \hookrightarrow C^{r_1}(\mathrm{End}(E))^2 \times C^{r_0}(\mathrm{End}(E))$$

and denote the resulting space by $\mathrm{Ell}^r(E)$. Equip $\overline{\mathrm{Ell}}(E)$ with the topology induced by the inclusion $\overline{\mathrm{Ell}}(E) \hookrightarrow \mathrm{Ell}^r(E) \times C^1(\mathrm{Gr}(E_\partial))$, with the product topology on the last space, and denote the resulting space by $\overline{\mathrm{Ell}}^r(E)$. The topologies on $\mathrm{Ell}^r(E)$ and $\overline{\mathrm{Ell}}^r(E)$ defined in such a way do not depend on the choice of a frame (e_1, e_2) and of a connection ∇.

By Proposition 15.2 the natural inclusion $\overline{\mathrm{Ell}}^{(0,0)}(E) \hookrightarrow \mathcal{R}^{sa}_F(L^2(E))$ is continuous. Since the (r_1, r_0)-topology on $\overline{\mathrm{Ell}}(E)$ is stronger than the $(0,0)$-topology, the inclusion $\overline{\mathrm{Ell}}^r(E) \hookrightarrow \mathcal{R}^{sa}_F(L^2(E))$ is continuous for every couple r of non-negative integers.

Convention. *From now on we will use the $(1,0)$-topology on $\overline{\mathrm{Ell}}(E)$, that is, the C^1-topology on symbols and the C^0-topology on free terms of operators. For brevity we will omit the superscript, so further $\overline{\mathrm{Ell}}(E)$ will always mean $\overline{\mathrm{Ell}}^{(1,0)}(E)$.*

The following proposition is an immediate corollary of Proposition 15.2.

Proposition 16.1. *The natural inclusion $\overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_F(L^2(E))$, $(A, L) \mapsto A_L$ is continuous.*

Remark 16.2. We choose to use the stronger $(1,0)$-topology on $\overline{\mathrm{Ell}}(E)$ instead of the $(0,0)$-topology to simplify the proofs. Probably, all theorems in the thesis remain valid for the $(0,0)$-topology on $\overline{\mathrm{Ell}}(E)$ as well, but the author did not check this. It can be easily seen that all universality results are valid (and their proofs remains the same) for the (r_1, r_0)-topology on $\overline{\mathrm{Ell}}(E)$ with $r_1 - 1 \geqslant r_0 \geqslant 0$.

Continuity of the decomposition. We prove here a technical result that will be used further in the thesis.

Denote by $\Sigma(E)$ the set of all smooth bundle morphisms $\sigma\colon T^*M \to \mathrm{End}(E)$ such that σ is a symbol of a formally self-adjoint elliptic operator. Equip $\Sigma(E)$ with the topology induced by the inclusion $\Sigma(E) \hookrightarrow C^1(TM \otimes \mathrm{End}(E))$. Then the natural projection $\mathrm{Ell}(E) \to \Sigma(E)$ is continuous, as well as the map $\Sigma(E) \to C^1(\mathrm{End}(E_\partial))$ taking σ to $\sigma(n)$.

For a smooth fiber bundle V over a smooth compact manifold N, we denote by $C^{\infty,s}(V)$ the space of smooth sections of V with the C^s-topology, that is, the topology induced by the embedding $C^\infty(V) \hookrightarrow C^s(V)$.

Let e_1, e_2 be global sections of T^*M such that $(e_1(x), e_2(x))$ is a positive oriented orthonormal basis of T^*_xM for any $x \in M$.

Proposition 16.3. *The following maps are continuous:*

1. $Q\colon \Sigma(E) \to C^{\infty,1}(\mathrm{End}(E))$, $\sigma \mapsto Q = \sigma(e_1)^{-1}\sigma(e_2)$;
2. $E^+, E^-\colon \Sigma(E) \to C^{\infty,1}(\mathrm{Gr}(E))$.

Proof. 1. The maps from $\Sigma(E)$ to $C^{\infty,1}(\mathrm{End}(E))$ taking σ to $\sigma(e_i)$, $i = 1, 2$, are continuous, so Q is also continuous.

2. The invariant subspace E^-_x of Q_x spanned by the generalized eigenspaces of Q_x corresponding to eigenvalues with negative imaginary part is an analytic function of Q_x and hence an analytic function of σ_x. Therefore, for smooth σ, $E^-(\sigma)$ is a smooth subbundle of E, and the map $E^-\colon \Sigma(E) \to C^{\infty,1}(\mathrm{Gr}(E))$ is continuous. The same is true for $E^+\colon \Sigma(E) \to C^{\infty,1}(\mathrm{Gr}(E))$.

Correspondence between L and T. Proposition 15.3 defines a one-to-one correspondence between L and T. We will use it to construct homotopies in $\Omega_g\overline{\mathrm{Ell}}(E)$. To do this, we need to show that the map $(A, L) \mapsto (A, T)$ is a homeomorphism.

Denote by $\overline{\mathrm{Ell}}'(E)$ the set of all pairs (A, T) such that $A \in \mathrm{Ell}(E)$ and T is a smooth bundle automorphism of $E^-_\partial(A)$. We equip $\overline{\mathrm{Ell}}'(E)$ with the topology induced by the inclusion

$$\overline{\mathrm{Ell}}'(E) \hookrightarrow \mathrm{Ell}(E) \times C^1(\mathrm{End}(E_\partial)), \quad (A, T) \mapsto (A, T \oplus \mathrm{Id}_{(E^-_\partial)^\perp}), \tag{16.1}$$

where $(E^-_\partial)^\perp$ is the orthogonal complement of $E^-_\partial(A)$ in E_∂. We introduce the auxiliary self-adjoint automorphism

$$T' = T \oplus \mathrm{Id}_{(E^-_\partial)^\perp} \tag{16.2}$$

by technical reasons: T acts on the bundle $E^-_\partial(A)$ which depends on A, while T' acts on the fixed bundle E_∂.

Proposition 16.4. *The map* $\overline{\mathrm{Ell}}(E) \to \overline{\mathrm{Ell}}'(E)$ *taking* (A, L) *to* (A, T) *is a homeomorphism. The map* $F\colon \overline{\mathrm{Ell}}(E) \to C^{\infty,1}(\mathrm{Gr}(E_\partial))$ *is continuous.*

Proof. Denote by $\mathrm{Gr}^{(2)}(E)$ the smooth subbundle of $\mathrm{Gr}(E) \times_M \mathrm{Gr}(E)$ whose fiber over $x \in M$ consists of pairs (V_x, W_x) of subspaces of E_x such that $V_x \cap W_x = 0$ and $V_x + W_x = E_x$. For a smooth section (V, W) of $\mathrm{Gr}^{(2)}(E)$ the projection $P_{V,W}$ of E on V along W is a smooth section of $\mathrm{End}(E)$. The map $C^{\infty,1}(\mathrm{Gr}^{(2)}(E)) \to C^{\infty,1}(\mathrm{End}(E))$, $(V, W) \to P_{V,W}$ is continuous. The same is true if we replace M by ∂M and E by E_∂. Therefore, the composition

$$\Sigma(E) \to C^{\infty,1}(\mathrm{Gr}^{(2)}(E)) \to C^{\infty,1}(\mathrm{Gr}^{(2)}(E_\partial)) \to C^{\infty,1}(\mathrm{End}(E_\partial)),$$

$\sigma \mapsto (E^+, E^-) \mapsto (E^+_\partial, E^-_\partial) \mapsto P_{E^+_\partial, E^-_\partial} = P^+(\sigma)$, is continuous. Similarly, the map $P^-\colon \Sigma(E) \to C^{\infty,1}(\mathrm{End}(E_\partial))$ is continuous.

Let T' be defined by formula (16.2). Since $TP^- = T'P^-$, identity (15.5) can be equivalently written as $P_T = P^+ + i\sigma(n)^{-1}(P^-)^*T'P^-$. Hence P_T, and also $L = \operatorname{Ker} P_T$, continuously depend on (σ, T'). It follows that the map $(A, T') \mapsto (A, L)$ is continuous.

Conversely, for $u \in E_\partial^-$ we have $Tu = P_{\mathrm{ort}}^- i\sigma(n)P^+P_{L,E_\partial^+}u$, where $P_{\mathrm{ort}}^- = P^-(P^- + (P^-)^* - 1)^{-1}$ is the orthogonal projection of E_∂ onto E_∂^- (see (9.3) for the formula of the orthogonal projection). This implies

$$T' = P_{\mathrm{ort}}^-\, i\sigma(n)\, P^+\, P_{L,E_\partial^+} P_{\mathrm{ort}}^- + (1 - P_{\mathrm{ort}}^-).$$

Since all elements of this expression continuously depend on (σ, L), the map $(A, L) \mapsto (A, T')$ is continuous. This proves the first part of the proposition.

By the definition of T', we have $\chi_{(-\infty,0)}(T_x) = \chi_{(-\infty,0)}(T'_x)$, where χ_S denotes the characteristic function of a subset S of $\mathbb{R}$. Hence F_x considered as a point of $\mathrm{Gr}(E_x)$ coincides with $\mathrm{Im}\,\big(\chi_{(-\infty,0)}(T'_x)\big)$ and thus is an analytic function of T'_x. Therefore, F is a smooth subbundle of E and continuously depends on T' in the C^1-topology. Together with the continuity of T' this implies continuity of the map $F\colon \overline{\mathrm{Ell}}(E) \to C^{\infty,1}(\mathrm{Gr}(E_\partial))$. This proves the second part of the proposition. $\square$

Part V

Spectral flow

17 The invariant Ψ and its properties

Gluing of bundles. Let $\gamma \in \Omega_g \overline{\mathrm{Ell}}(E)$, that is, $\gamma\colon [0,1] \to \overline{\mathrm{Ell}}(E)$ is a path in $\overline{\mathrm{Ell}}(E)$ such that $\gamma(1) = g\gamma(0)$, $g \in U(E)$. With every such pair (γ, g) we associate a number of vector bundles.

First, lift E to the vector bundle $\widehat{E} = E \times [0,1]$ over $M \times [0,1]$. Then form the vector bundle $\mathcal{E}$ over $M \times S^1$ as the factor of $\widehat{E}$, identifying $(u, 1)$ with $(gu, 0)$ for every $u \in E$.

The one-parameter family $E_t^- = E^-(\gamma(t))$ of subbundles of E forms the subbundle $\widehat{E}^-$ of $\widehat{E}$. The condition $\gamma(1) = g\gamma(0)$ implies $E_1^- = gE_0^-$, so $\widehat{E}^-$ descends onto $M \times S^1$ giving rise to the subbundle $\mathcal{E}^- = \mathcal{E}^-(\gamma, g)$ of $\mathcal{E}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \widehat{E}^- & \hookrightarrow & \widehat{E} & \longrightarrow & M \times [0,1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}^- & \hookrightarrow & \mathcal{E} & \longrightarrow & M \times S^1 \end{array}$$

In the same manner, from the one-parameter family of vector bundles $E_\partial^-(\gamma(t)) \subset E_\partial$ we construct the vector bundles $\widehat{E}_\partial^- \subset \widehat{E}_\partial$ over $\partial M \times [0,1]$. Twisting by g and gluing as described above, we obtain the vector bundles $\mathcal{E}_\partial^- \subset \mathcal{E}_\partial$ over $\partial M \times S^1$. Equivalently, $\mathcal{E}_\partial$ and $\mathcal{E}_\partial^-$ can be obtained as the restrictions of $\mathcal{E}$ and $\mathcal{E}^-$ to $\partial M \times S^1$.

The one-parameter family $F_t = F(\gamma(t))$ of subbundles of $E_\partial^-(\gamma(t))$ forms the subbundle $\widehat{F}$ of $\widehat{E}_\partial^-$. Again, the condition $\gamma(1) = g\gamma(0)$ implies $F_1 = gF_0$, so $\widehat{F}$ descends onto $\partial M \times S^1$ giving rise to the subbundle $\mathcal{F} = \mathcal{F}(\gamma, g)$ of $\mathcal{E}_\partial^-$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} \widehat{F} & \hookrightarrow & \widehat{E}_\partial^- & \hookrightarrow & \widehat{E}_\partial & \longrightarrow & \partial M \times [0,1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{F} & \hookrightarrow & \mathcal{E}_\partial^- & \hookrightarrow & \mathcal{E}_\partial & \longrightarrow & \partial M \times S^1 \end{array}$$

If $g = \mathrm{Id}$, then we will write $\mathcal{F}(\gamma)$ instead of $\mathcal{F}(\gamma, \mathrm{Id})$.

Definition of $\Psi(\gamma, g)$. The orientation on M induces the orientation on ∂M. We equip ∂M with an orientation in such a way that the pair (outward normal to ∂M, positive tangent vector to ∂M) has a positive orientation.

The product $\partial M \times S^1$ is a two-dimensional manifold, namely a disjoint union of tori. Let $[\partial M \times S^1] \in H_2(\partial M \times S^1)$ be its fundamental class. The first Chern class $c_1(\mathcal{F})$ of the vector bundle $\mathcal{F}$ is an element of the second cohomology group $H^2(\partial M \times S^1)$, so one can compute its value on $[\partial M \times S^1]$, obtaining the integer-valued invariant

$$\Psi(\gamma, g) = c_1(\mathcal{F}(\gamma, g))[\partial M \times S^1]. \qquad (17.1)$$

If $g = \mathrm{Id}$, then we will write $\Psi(\gamma)$ instead of $\Psi(\gamma, \mathrm{Id})$.

The homomorphism ψ. The first Chern class is additive with respect to direct sum of vector bundles, so we can define the homomorphism of commutative groups $\psi\colon K^0(\partial M \times S^1) \to \mathbb{Z}$ by the rule $\psi[V] = c_1(V)[\partial M \times S^1]$ for any vector bundle V over $\partial M \times S^1$. Then Ψ can be written as

$$\Psi(\gamma, g) = \psi[\mathcal{F}(\gamma, g)].$$

Consider the following three subgroups of $K^0(\partial M \times S^1)$:

- G^* is the image of the natural homomorphism $K^0(\partial M) \to K^0(\partial M \times S^1)$ induced by the projection $\partial M \times S^1 \to \partial M$.
- G^∂ is the image of the homomorphism $K^0(M \times S^1) \to K^0(\partial M \times S^1)$ induced by the embedding $\partial M \times S^1 \hookrightarrow M \times S^1$.
- G is the subgroup of $K^0(\partial M \times S^1)$ spanned by G^* and G^∂.

Proposition 17.1. *The homomorphism ψ is surjective with the kernel G. In other words, the following sequence is exact:*

$$0 \longrightarrow G \longrightarrow K^0(\partial M \times S^1) \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0.$$

Proof. Denote the connected components of ∂M by $\partial M_1, \ldots, \partial M_m$. The group $K^0(\partial M \times S^1)$ is isomorphic to $\mathbb{Z}^{2m}$, with the isomorphism given by

$$[V] \mapsto (r_1, \ldots, r_m, a_1, \ldots, a_m),$$

where r_j is the rank of the restriction V_j of a vector bundle V to $\partial M_j \times S^1$ and $a_j = c_1(V_j)[\partial M_j \times S^1]$.

In these designations, the subgroup G^* consists of elements with $a_1 = \ldots = a_m = 0$. The subgroup G^∂ consists of elements with $r_1 = \ldots = r_m$ and $\sum_j a_j = 0$. The span G of G^* and G^∂ consists of elements with $\sum_j a_j = 0$. The homomorphism ψ takes $(r_1, \ldots, r_m, a_1, \ldots, a_m)$ to $\sum_j a_j$, so it is surjective with the kernel G. This completes the proof of the proposition. □

Special subspaces. The following two subspaces of $\overline{\mathrm{Ell}}(E)$ will play a special role:

- $\overline{\mathrm{Ell}}^+(E)$ consists of all $(A, T) \in \overline{\mathrm{Ell}}(E)$ with positive definite T.
- $\overline{\mathrm{Ell}}^-(E)$ consists of all $(A, T) \in \overline{\mathrm{Ell}}(E)$ with negative definite T.

Proposition 17.2. *Let $\gamma \in \Omega_g\overline{\mathrm{Ell}}(E)$. Then the following statements hold:*

1. $\mathcal{F}(\gamma, g) = 0$ *if and only if* $\gamma \in \Omega_g\overline{\mathrm{Ell}}^+(E)$;
2. $\mathcal{F}(\gamma, g) = \mathcal{E}_\partial^-(\gamma, g)$ *if and only if* $\gamma \in \Omega_g\overline{\mathrm{Ell}}^-(E)$.

Proof. It follows immediately from the definition of $\mathcal{F}$. □

Properties of Ψ. Denote by $\Omega^*\overline{\mathrm{Ell}}(E)$ the subspace of $\Omega\overline{\mathrm{Ell}}(E)$ consisting of constant loops.

Proposition 17.3. Ψ *has the following properties:*

(Ψ0) Ψ *vanishes on* $\Omega^*\overline{\mathrm{Ell}}(E)$, $\Omega_g\overline{\mathrm{Ell}}^+(E)$, *and* $\Omega_g\overline{\mathrm{Ell}}^-(E)$ *for every* $g \in U(E)$.

(Ψ1) Ψ *is constant on path connected components of* $\Omega_g\overline{\mathrm{Ell}}(E)$ *for every* $g \in U(E)$.

(Ψ2) $\Psi(\gamma_0 \oplus \gamma_1, g_0 \oplus g_1) = \Psi(\gamma_0, g_0) + \Psi(\gamma_1, g_1)$ *for* $\gamma_i \in \Omega_{g_i}\overline{\mathrm{Ell}}(E_i)$, $g_i \in U(E_i)$, $i = 0, 1$.

Proof. (Ψ0). If $\gamma \in \Omega_g\overline{\mathrm{Ell}}^+(E)$, then $\mathcal{F}(\gamma, g) = 0$, so $\Psi(\gamma, g) = 0$.

If $\gamma \in \Omega_g\overline{\mathrm{Ell}}^-(E)$, then $\mathcal{F}(\gamma, g)$ is the restriction to $\partial M \times S^1$ of the vector bundle $\mathcal{E}^-(\gamma, g)$ over $M \times S^1$, so $[\mathcal{F}(\gamma, g)] \in G^\partial$.

If $\gamma \in \Omega^*\overline{\mathrm{Ell}}(E)$, $\gamma(t) \equiv (A, L)$, then $\mathcal{F}(\gamma)$ is the lifting to $\partial M \times S^1$ of the vector bundle $F(A, L)$ over ∂M, so $[\mathcal{F}(\gamma)] \in G^*$.

In the last two cases Proposition 17.1 implies vanishing of Ψ.

(Ψ1). If γ_0 and γ_1 are connected by a path (γ_s) in $\Omega_g\overline{\mathrm{Ell}}(2k_M)$, then $\mathcal{F}_0 = \mathcal{F}(\gamma_0, g)$ and $\mathcal{F}_1 = \mathcal{F}(\gamma_1, g)$ are homotopic via the homotopy $s \mapsto \mathcal{F}(\gamma_s, g)$. It follows that the classes of $\mathcal{F}_0$ and $\mathcal{F}_1$ in $K^0(\partial M \times S^1)$ coincide, and thus $\Psi(\gamma_0, g) = \psi[\mathcal{F}_0] = \psi[\mathcal{F}_1] = \Psi(\gamma_1, g)$.

(Ψ2). Obviously, $\mathcal{F}(\gamma_0 \oplus \gamma_1, g_0 \oplus g_1) = \mathcal{F}(\gamma_0, g_0) \oplus \mathcal{F}(\gamma_1, g_1)$. Passing to the classes in $K^0(\partial M \times S^1)$ and applying the homomorphism ψ, we obtain the additivity of Ψ. □

18 Dirac operators

Odd Dirac operators. Recall that $A \in \mathrm{Ell}(E)$ is called a Dirac operator if $\sigma_A(\xi)^2 = \|\xi\|^2 \,\mathrm{Id}_E$ for all $\xi \in T^*M$. We denote by $\mathrm{Dir}(E)$ the subspace of $\mathrm{Ell}(E)$ consisting of all *odd* Dirac operators, that is, operators having the form

(18.1)
$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix} \text{ with respect to the chiral decomposition } E = E^+(A) \oplus E^-(A).$$

Denote by $\overline{\mathrm{Dir}}(E)$ the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that $A \in \mathrm{Dir}(E)$.

The following two subspaces of $\overline{\mathrm{Dir}}(E)$ will play a special role:

$$\overline{\mathrm{Dir}}^+(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^+(E), \quad \overline{\mathrm{Dir}}^-(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^-(E).$$

Realization of bundles. In the following we will need the possibility to realize some vector bundles over $\partial M \times S^1$ as $\mathcal{F}(\gamma)$ for some γ. Recall that we denoted by k_N the trivial vector bundle of rank k over N.

Proposition 18.1. *Every smooth vector bundle* V *over* ∂M *can be realized as* $F(A, L)$ *for some* $k \in \mathbb{N}$ *and* $(A, L) \in \overline{\mathrm{Dir}}(2k_M)$.

Proof. V can be embedded as a smooth subbundle in a trivial vector bundle $k_{\partial M}$ of sufficient large rank k. Choose a smooth global field (e_1, e_2) of positive oriented orthonormal frames of TM and define the Dirac operator acting on sections of k_M (that is, $\mathbb{C}^k$-valued functions on M) by the formula $D = -i\partial_1 + \partial_2$. Let D^t be the operator formally adjoint to D. Then

$$A = \begin{pmatrix} 0 & D^t \\ D & 0 \end{pmatrix} \tag{18.2}$$

is an odd Dirac operator acting on sections of $k_M \oplus k_M$, and $E^-(A) = k_M$. Let $V^\perp$ be the orthogonal complement of V in $E^-_\partial(A) = k_{\partial M}$, and let L be the boundary condition for A defined by $T = (-1)_V \oplus 1_{V^\perp}$. Then $(A, L) \in \overline{\mathrm{Dir}}(2k_M)$ and $F(A, L) = V$, which proves the proposition. □

Proposition 18.2. *Every smooth vector bundle V over $\partial M \times S^1$ can be realized as $\mathcal{F}(\gamma)$ for some $k \in \mathbb{N}$ and $\gamma \in \Omega\,\overline{\mathrm{Dir}}(2k_M)$.*

Proof. V can be embedded as a smooth subbundle in a trivial vector bundle over $\partial M \times S^1$ of sufficient large rank k. Let (V_t), $t \in S^1$, be the corresponding one-parameter family of subbundles of $k_{\partial M}$. Define the odd Dirac operator $A \in \mathrm{Dir}(k_M \oplus k_M)$ by formula (18.2). Let L_t be the boundary condition for A corresponding to the automorphism $T = (-1)_{V_t} \oplus 1_{V_t^\perp}$ of k_M. The element $(A, L_t) \in \overline{\mathrm{Dir}}(2k_M)$ depends continuously on t, so the family (A, L_t) defines the loop $\gamma \in \Omega\,\overline{\mathrm{Dir}}(2k_M)$. By construction, $F(A, L_t) = V_t$, so $\mathcal{F}(\gamma) = V$, which completes the proof of the proposition. □

Proposition 18.3. *Let V be a smooth vector bundle over $M \times S^1$. Then the restriction V_∂ of V to $\partial M \times S^1$ can be realized as $\mathcal{F}(\gamma, g)$ for some $\gamma \in \Omega_g\overline{\mathrm{Dir}}^-(2k_M)$, $k \in \mathbb{N}$, $g \in U(2k_M)$.*

Proof. Let k be the rank of V. The lifting of V by the map $M \times [0,1] \to M \times S^1$ is a trivial vector bundle $k_{M\times[0,1]}$, so we can obtain V from this trivial bundle, gluing $k_{M\times\{1\}}$ with $k_{M\times\{0\}}$ by some unitary bundle automorphism $g \in U(k_M)$. Let $E = k_M \oplus k_M$, $\tilde{g} = g \oplus g \in U(E)$, and $A \in \mathrm{Dir}(E)$ be given by formula (18.2). Since the symbol of A is $\tilde{g}$-invariant, $A_1 = \tilde{g}A\tilde{g}^{-1}$ has the same symbol as A, so the path $[0,1] \ni t \mapsto A_t = (1-t)A + tA_1$ is an element of $\Omega_{\tilde{g}}\,\mathrm{Dir}(E)$. It follows that the path γ given by the formula $\gamma(t) = (A_t, -\mathrm{Id})$ is an element of $\Omega_{\tilde{g}}\overline{\mathrm{Dir}}^-(E)$. By construction, $\mathcal{F}(\gamma, g) = V_\partial$. This completes the proof of the proposition. □

Proposition 18.4. *Every integer λ can be obtained as $\lambda = \Psi(\gamma)$ for some $k \in \mathbb{N}$ and $\gamma \in \Omega\,\overline{\mathrm{Dir}}(2k_M)$.*

Proof. Every integer λ can be obtained as the first Chern number of a smooth vector bundle over a torus. Hence $\lambda = \psi[V]$ for some smooth vector bundle V over $\partial M \times S^1$. By Proposition 18.2 V can be realized as $V = \mathcal{F}(\gamma)$ for some $\gamma \in \Omega\,\overline{\mathrm{Dir}}_M$. We obtain $\lambda = \Psi(\gamma)$, which completes the proof of the proposition. □

19 Universality of Ψ

Homotopies fixing the operators. In this section we will deal only with such deformations of elements of $\Omega_g\overline{\mathrm{Ell}}(E)$ that fix an operator family $\mathcal{A} = (A_t)$ and change only boundary conditions (L_t).

Let us fix an odd Dirac operator $D \in \mathrm{Dir}(2_M)$. Denote by $\delta^+ \in \Omega^*\overline{\mathrm{Dir}}^+(2_M)$, respectively $\delta^- \in \Omega^*\overline{\mathrm{Dir}}^-(2_M)$ the constant loop taking the value (D, Id), respectively $(D, -\mathrm{Id})$. We denote by $k\delta^+$, respectively $k\delta^-$ the direct sum of k copies of δ^+, respectively δ^-. Notice that $\mathcal{E}_\partial^-(k\delta^+) = \mathcal{E}_\partial^-(k\delta^-) = \mathcal{F}(k\delta^-) = k_{\partial M\times S^1}$ and $\mathcal{F}(k\delta^+) = 0$.

Proposition 19.1. *Let $\gamma\colon t \mapsto (A_t, L_t)$ and $\gamma'\colon t \mapsto (A_t, L_t')$, $t \in [0,1]$, be elements of $\Omega_g\overline{\mathrm{Ell}}(E)$ differing only by boundary conditions. Then the following statements hold.*

1. *If $\mathcal{F}(\gamma, g)$ and $\mathcal{F}(\gamma', g)$ are homotopic subbundles of $\mathcal{E}_\partial^-(\gamma, g)$, then γ and γ' can be connected by a path in $\Omega_g\overline{\mathrm{Ell}}(E)$.*
2. *If $\mathcal{F}(\gamma, g)$ and $\mathcal{F}(\gamma', g)$ are isomorphic as vector bundles, then $\gamma \oplus k\delta^+$ and $\gamma' \oplus k\delta^+$ can be connected by a path in $\Omega_{g\oplus\mathrm{Id}}\overline{\mathrm{Ell}}(E \oplus 2k_M)$ for k large enough.*
3. *If $[\mathcal{F}(\gamma, g)] = [\mathcal{F}(\gamma', g)] \in K^0(\partial M \times S^1)$, then $\gamma \oplus l\delta^- \oplus k\delta^+$ and $\gamma' \oplus l\delta^- \oplus k\delta^+$ can be connected by a path in $\Omega_{g\oplus\mathrm{Id}\oplus\mathrm{Id}}\overline{\mathrm{Ell}}(E \oplus 2l_M \oplus 2k_M)$ for l, k large enough.*

Proof. Notice that $\mathcal{E}_\partial^-(\gamma, g)$ depends only on the operators and does not depend on the boundary conditions, so $\mathcal{E}_\partial^-(\gamma, g) = \mathcal{E}_\partial^-(\gamma', g)$. Denote $\mathcal{E}_\partial^- = \mathcal{E}_\partial^-(\gamma, g)$, $\mathcal{F} = \mathcal{F}(\gamma, g)$, and $\mathcal{F}' = \mathcal{F}(\gamma', g)$.

1. Let $\mathcal{A} = (A_t) \in \Omega_g \mathrm{Ell}(E)$ be the corresponding path of operators. Denote by $\mathcal{L}(\mathcal{A}, g)$ the space of all lifts of $\mathcal{A}$ to $\Omega_g\overline{\mathrm{Ell}}(E)$. Denote by $\mathcal{L}^u(\mathcal{A}, g)$ the subspace of $\mathcal{L}(\mathcal{A}, g)$ consisting of paths (A_t, T_t) such that the self-adjoint automorphism T_t is unitary for every $t \in [0,1]$. The subspace $\mathcal{L}^u(\mathcal{A}, g)$ is a strong deformation retract of $\mathcal{L}(\mathcal{A}, g)$, with the retraction given by the formula $q_s(A_t, T_t) = (A_t, (1 - s + s|T_t|^{-1})T_t)$. Since q_s preserves $\mathcal{F}$, it is sufficient to prove the first claim of the proposition for $\gamma, \gamma' \in \mathcal{L}^u(\mathcal{A}, g)$.

For a fixed $\mathcal{A}$, an element $\gamma \in \mathcal{L}^u(\mathcal{A}, g)$ is uniquely defined by a subbundle $\mathcal{F}(\gamma, g)$ of $\mathcal{E}_\partial^-(\gamma, g)$, and every deformation of $\mathcal{F}$ uniquely defines the deformation of γ. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are homotopic subbundles of $\mathcal{E}_\partial^-$. A homotopy h_s between $\mathcal{F}$ and $\mathcal{F}'$ can be chosen to be smooth by $x \in \partial M$ and continuous (in the C^1-topology) by $s, t \in [0,1]$. As described above, such a homotopy defines a path connecting γ and γ' in $\mathcal{L}^u(\mathcal{A}, g) \subset \Omega_g\overline{\mathrm{Ell}}(E)$. This completes the proof of the first claim of the proposition.

2. If $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as vector bundles, then $\mathcal{F} \oplus 0$ and $\mathcal{F}' \oplus 0$ are homotopic as subbundles of $\mathcal{E}_\partial^- \oplus k_{\partial M\times S^1}$ for k large enough. It remains to apply the first part of the proposition to the elements $\gamma \oplus k\delta^+$ and $\gamma' \oplus k\delta^+$ of $\Omega_{g\oplus\mathrm{Id}}\overline{\mathrm{Ell}}(E \oplus 2k_M)$.

3. The equality $[\mathcal{F}] = [\mathcal{F}']$ implies that the vector bundles $\mathcal{F}$ and $\mathcal{F}'$ are stably isomorphic, that is, $\mathcal{F}_1 \oplus l_{\partial M\times S^1}$ and $\mathcal{F}_2 \oplus l_{\partial M\times S^1}$ are isomorphic for some integer l. It remains to apply the second part of the proposition to the elements $\gamma \oplus l\delta^-$ and $\gamma' \oplus l\delta^-$ of $\Omega_{g\oplus\mathrm{Id}}\overline{\mathrm{Ell}}(E \oplus 2l_M)$. □

The case of different operators. For $\gamma \in \Omega_g \overline{\mathrm{Ell}}(E)$, $\gamma(t) = (A_t, T_t)$, we denote by γ^+ the element of $\Omega_g \overline{\mathrm{Ell}}^+(E)$ given by the rule $t \mapsto (A_t, \mathrm{Id})$.

Let $\gamma_i \in \Omega_{g_i} \overline{\mathrm{Ell}}(E_i)$, $i = 1, 2$. Consider the elements $\gamma_1' = \gamma_1 \oplus \gamma_2^+$ and $\gamma_2' = \gamma_1^+ \oplus \gamma_2$ of $\Omega_{g_1 \oplus g_2} \overline{\mathrm{Ell}}(E_1 \oplus E_2)$. By Proposition 17.2 $\mathcal{F}(\gamma_i', g_1 \oplus g_2) = \mathcal{F}(\gamma_i, g_i)$. On the other hand, γ_1' and γ_2' differ only by boundary conditions and thus fall within the framework of Proposition 19.1. In particular, from the third part of Proposition 19.1 we immediately get the following.

Proposition 19.2. *Let $\gamma_i \in \Omega_{g_i} \overline{\mathrm{Ell}}(E_i)$, $i = 1, 2$. Suppose that $[\mathcal{F}(\gamma_1, g_1)] = [\mathcal{F}(\gamma_2, g_2)] \in K^0(\partial M \times S^1)$. Then*

$$\gamma_1 \oplus \gamma_2^+ \oplus l\delta^- \oplus k\delta^+ \ \text{ and } \ \gamma_1^+ \oplus \gamma_2 \oplus l\delta^- \oplus k\delta^+$$

can be connected by a path in $\Omega_{g_1 \oplus g_2 \oplus \mathrm{Id} \oplus \mathrm{Id}} \overline{\mathrm{Ell}}(E_1 \oplus E_2 \oplus 2l_M \oplus 2k_M)$ if l, k are large enough.

Semigroup of elliptic operators. The disjoint union

$$\overline{\mathrm{Ell}}_M = \coprod_{k \in \mathbb{N}} \overline{\mathrm{Ell}}(2k_M)$$

has the natural structure of a (non-commutative) graded topological semigroup with respect to the direct sum of operators and boundary conditions.

The point-wise direct sum of paths defines the map

$$\Omega_g \overline{\mathrm{Ell}}(2k_M) \times \Omega_{g'} \overline{\mathrm{Ell}}(2k'_M) \to \Omega_{g \oplus g'} \overline{\mathrm{Ell}}(2(k+k')_M),$$

which induces the natural structure of a (non-commutative) topological semigroup on the disjoint union

$$\Omega_U \overline{\mathrm{Ell}}_M = \coprod_{k \in \mathbb{N},\, g \in U(2k_M)} \Omega_g \overline{\mathrm{Ell}}(2k_M) = \left\{ (\gamma, g) \colon \gamma \in \Omega_g \overline{\mathrm{Ell}}(2k_M), k \in \mathbb{N},\ g \in U(2k_M) \right\}.$$

The disjoint unions

$$\Omega_U \overline{\mathrm{Ell}}_M^+ = \coprod_{k,g} \Omega_g \overline{\mathrm{Ell}}^+(2k_M), \ \Omega_U \overline{\mathrm{Ell}}_M^- = \coprod_{k,g} \Omega_g \overline{\mathrm{Ell}}^-(2k_M), \text{ and } \Omega^* \overline{\mathrm{Ell}}_M = \coprod_k \Omega^* \overline{\mathrm{Ell}}(2k_M)$$

are subsemigroups of $\Omega_U \overline{\mathrm{Ell}}_M$.

Universality of Ψ. Now we are ready to state the main result of this section.

Theorem 19.3. *Let Φ be a semigroup homomorphism from $\Omega_U \overline{\mathrm{Ell}}_M$ to a commutative monoid Λ, which is constant on path connected components of $\Omega_U \overline{\mathrm{Ell}}_M$. Then the following two conditions are equivalent:*

1. *Φ vanishes on $\Omega^* \overline{\mathrm{Ell}}_M$, $\Omega_U \overline{\mathrm{Ell}}_M^+$, and $\Omega_U \overline{\mathrm{Ell}}_M^-$.*

2. $\Phi = \vartheta \circ \Psi$ *for some (unique) monoid homomorphism* $\vartheta\colon \mathbb{Z} \to \Lambda$*, that is,* Φ *has the form* $\Phi(\gamma, g) = c \cdot \Psi(\gamma, g)$ *for some invertible constant* $c \in \Lambda$.

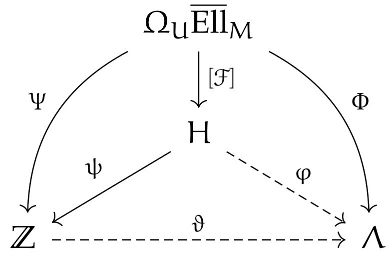
Here by "invertible" we mean that there is $c' \in \Lambda$ inverse to c, that is, such that $c' + c = 0$.

Proof. ($2 \Rightarrow 1$) follows immediately from properties ($\Psi 0$–$\Psi 2$) of Proposition 17.3.

Let us prove ($1 \Rightarrow 2$). Suppose that Φ satisfies condition (1) of the theorem. By Proposition 19.2 the equality $[\mathcal{F}(\gamma_1, g_1)] = [\mathcal{F}(\gamma_2, g_2)]$ implies

$$\Phi(\gamma_1 \oplus \gamma_2^+ \oplus l\delta^- \oplus k\delta^+, g_1 \oplus g_2 \oplus \mathrm{Id}) = \Phi(\gamma_1^+ \oplus \gamma_2 \oplus l\delta^- \oplus k\delta^+, g_1 \oplus g_2 \oplus \mathrm{Id}). \tag{19.1}$$

Since Φ vanishes on $(\gamma_i^+, g_i) \in \Omega_U\overline{\mathrm{Ell}}_M^+$, $(\delta^+, \mathrm{Id}) \in \Omega_U\overline{\mathrm{Ell}}_M^+$, and $(\delta^-, \mathrm{Id}) \in \Omega_U\overline{\mathrm{Ell}}_M^-$, (19.1) implies $\Phi(\gamma_1, g_1) = \Phi(\gamma_2, g_2)$. It follows that the homomorphism $\Phi\colon \Omega_U\overline{\mathrm{Ell}}_M \to \Lambda$ factors through the (unique) semigroup homomorphism $\varphi\colon H \to \Lambda$, where H denotes the image of $\Omega_U\overline{\mathrm{Ell}}_M$ in $K^0(\partial M \times S^1)$:



Suppose that $\psi(h_1) = \psi(h_2)$ for $h_1, h_2 \in H$. By Proposition 17.1 this implies $h_1 - h_2 = \mu^* + \mu^\partial \in K^0(\partial M \times S^1)$ for some $\mu^* \in G^*$ and $\mu^\partial \in G^\partial$.

The element μ^∂ can be written as the difference of classes $[j^*V_2] - [j^*V_1]$ for some (smooth) vector bundles V_1, V_2 over $M \times S^1$, where j denotes the embedding $\partial M \times S^1 \hookrightarrow M \times S^1$. By Proposition 18.3, $[j^*V_i]$ can be realized as $[\mathcal{F}(\beta_i, g_i')]$ for some $(\beta_i, g_i') \in \Omega_U\overline{\mathrm{Dir}}_M^-$, which gives $\mu^\partial = [\mathcal{F}(\beta_2, g_2')] - [\mathcal{F}(\beta_1, g_1')]$.

Similarly, by Proposition 18.1 $\mu^* = [\mathcal{F}(\alpha_2)] - [\mathcal{F}(\alpha_1)]$ for some $\alpha_1, \alpha_2 \in \Omega^*\,\overline{\mathrm{Dir}}_M$.

Combining all this, for liftings (γ_i, g_i) of h_i to $\Omega_U\overline{\mathrm{Ell}}_M$ we obtain

$$[\mathcal{F}(\gamma_1, g_1)] + [\mathcal{F}(\beta_1, g_1')] + [\mathcal{F}(\alpha_1)] = [\mathcal{F}(\gamma_2, g_2)] + [\mathcal{F}(\beta_2, g_2')] + [\mathcal{F}(\alpha_2)],$$

Applying φ to the both sides of this equality and taking into account that

$$\varphi\left([\mathcal{F}(\gamma_i, g_i)] + [\mathcal{F}(\beta_i, g_i')] + [\mathcal{F}(\alpha_i)]\right) = \Phi(\gamma_i, g_i) + \Phi(\beta_i, g_i') + \Phi(\alpha_i, \mathrm{Id}) = \Phi(\gamma_i, g_i) = \varphi(h_i),$$

we obtain $\varphi(h_1) = \varphi(h_2)$. Thus the equality $\psi(h_1) = \psi(h_2)$ implies $\varphi(h_1) = \varphi(h_2)$. On the other hand, the homomorphism $\psi\colon H \to \mathbb{Z}$ is surjective by Proposition 18.4. It follows that φ factors through the (unique) semigroup homomorphism $\vartheta\colon \mathbb{Z} \to \Lambda$. Since $\vartheta(0) = \Phi(\Omega^*\,\overline{\mathrm{Dir}}_M) = 0$, ϑ is a homomorphism of monoids.

Let $c = \vartheta(1)$ and $c' = \vartheta(-1)$. Then $c + c' = 0$ and $\vartheta(n) = nc$ for every $n \in \mathbb{Z}$. This completes the proof of the theorem. $\square$

20 Deformation retraction

The main result of this section is Proposition 20.5, where we prove that the natural embedding $\overline{\mathrm{Dir}}(E) \hookrightarrow \overline{\mathrm{Ell}}(E)$ is a homotopy equivalence. In the rest of Part V we will need only one corollary of this result, namely that every element of $\Omega_g\overline{\mathrm{Ell}}^+(E)$, respectively $\Omega_g\overline{\mathrm{Ell}}^-(E)$ is connected by a path with an element of $\Omega_g\overline{\mathrm{Dir}}^+(E)$, respectively $\Omega_g\overline{\mathrm{Dir}}^-(E)$.

Sections. First we construct two sections, which will be used below for construction of a deformation retraction.

Proposition 20.1. *The map* $p\colon \mathrm{Ell}(E) \to \Sigma(E)$ *is surjective and has a continuous section* $r\colon \Sigma(E) \to \mathrm{Ell}(E)$ *such that* $r \circ p$ *is fiberwise homotopic to the identity map.*

Proof. We define a section $r\colon \Sigma(E) \to \mathrm{Ell}(E)$ by the formula $r(\sigma) = (\sigma_1\nabla_1 + \sigma_2\nabla_2)/2 + (\sigma_1\nabla_1 + \sigma_2\nabla_2)^t/2$, where $\sigma_i = \sigma(e_i)$, (e_1, e_2) is a fixed global field of frames in TM, ∇ is a fixed smooth connection on E, and superscript t means taking of formally adjoint operator. The operation of taking formally adjoint operator leaves invariant symbol. Moreover, it defines a continuous transformation of the space of first order operators with the topology defined by the inclusion to $C^1(\mathrm{End}(E))^2 \times C^0(\mathrm{End}(E))$, $\sigma_1\nabla_1 + \sigma_2\nabla_2 + a \mapsto (\sigma_1, \sigma_2, a)$. Thus r is a continuous section of p and defines a trivialization of the affine bundle $\mathrm{Ell}(E) \to \Sigma(E)$ with the fiber $C^{\infty,0}(\mathrm{End}^{sa}(E))$. Thus $r \circ p$ is fiberwise homotopic to the identity map, which completes the proof of the proposition. □

Denote by $\Sigma^D(E) = p(\mathrm{Dir}(E))$ the subspace of $\Sigma(E)$ consisting of symbols of Dirac operators.

Proposition 20.2. *The restriction of* p *to* $\mathrm{Dir}(E)$ *has a continuous section* $r^D\colon \Sigma^D(E) \to \mathrm{Dir}(E)$.

Proof. Let $\sigma \in \Sigma^D(E)$ and $A = r(\sigma)$. Denote by S the bundle automorphism of E, whose restrictions on fibers are the orthogonal reflections in the fibers of $E^-(\sigma)$. We define $r^D(\sigma)$ by the formula $r^D(\sigma) = (A - SAS)/2$. Obviously, it is a Dirac operator, which is odd with respect to the chiral decomposition $E = E^+(\sigma) \oplus E^-(\sigma)$ and has the same symbol σ as A. Since S depends continuously on σ, the map $r^D\colon \Sigma^D(E) \to \mathrm{Dir}(E)$ is a continuous section of $p|_{\mathrm{Dir}(E)}$. This completes the proof of the proposition. □

Retraction of symbols. The following proposition is the key result of this section.

Proposition 20.3. *The subspace* $\Sigma^D(E)$ *is a strong deformation retract of* $\Sigma(E)$. *Moreover, a deformation retraction can be chosen to be* $U(E)$*-equivariant and to preserve* $E^-(\sigma)$.

Proof. For any $\sigma \in \Sigma(E)$ the automorphism $Q = \sigma(e_1)^{-1}\sigma(e_2)$ of E leaves the subbundles $E^- = E^-(\sigma)$ and $E^+ = E^+(\sigma)$ invariant. Denote by Q^- (respectively Q^+) the restriction of Q to E^- (respectively E^+). By the construction of E^- and E^+, all eigenvalues of Q_x^- (respectively Q_x^+) have negative (respectively positive) imaginary part for every $x \in M$.

Denote by J the restriction of $\sigma(e_1)$ to E^-; it is a smooth bundle isomorphism from E^- onto its orthogonal complement $(E^-)^\perp$.

Finally, with every $\sigma \in \Sigma(E)$ we associate the quadruple

$$\vartheta(\sigma) = (E^-, E^+, J, Q^-). \tag{20.1}$$

Denote by $\Theta(E)$ the set of all quadruples (E^-, E^+, J, Q^-) such that E^-, E^+ are transversal smooth subbundles of E of half rank (that is, $\operatorname{rank} E^- = \operatorname{rank} E^+ = \frac{1}{2} \operatorname{rank} E$), J is a smooth bundle isomorphism of E^- onto $(E^-)^\perp$, and Q^- is a smooth bundle automorphism of E^- such that all eigenvalues of Q^-_x have negative imaginary part for every $x \in M$.

Equip $\Theta(E)$ with the topology induced by the inclusion

$$\Theta(E) \hookrightarrow C^1(\mathrm{Gr}(E))^2 \times C^1(\mathrm{End}(E))^2, \quad (E^-, E^+, J, Q^-) \mapsto (E^-, E^+, J \oplus 0_{E^+}, Q^- \oplus 0_{E^+}).$$

Lemma 20.4. *The map* (20.1) *defines a homeomorphism between the spaces* $\Sigma(E)$ *and* $\Theta(E)$.

Proof. Let us show first that ϑ is a bijection. Let $(E^-, E^+, J, Q^-) \in \Theta(E)$. Then $\sigma_1^- = J$, $\sigma_2^- = JQ^-$ are smooth bundle isomorphisms from E^- onto $(E^-)^\perp$.

The Hermitian structure on E defines the non-degenerate pairings $E_x^+ \times (E_x^-)^\perp \to \mathbb{C}$ and $(E_x^+)^\perp \times E_x^- \to \mathbb{C}$ for each $x \in M$. Hence there exist (unique) smooth bundle isomorphisms σ_1^+, σ_2^+ from E^+ onto $(E^+)^\perp$ such that $\langle \sigma_i^+ u, v \rangle = \langle u, \sigma_i^- v \rangle$ for any $u \in E_x^+$, $v \in E_x^-$, $x \in M$. We define the endomorphism σ_i of E by the condition that the restriction of σ_i to E^+, respectively E^- coincides with σ_i^+, respectively σ_i^-.

Every elements $u, v \in E_x$ can be written as $u = u^+ + u^-$, $v = v^+ + v^-$ with $u^+, v^+ \in E_x^+$, $u^-, v^- \in E_x^-$. We get $\langle \sigma_i u, v \rangle = \langle \sigma_i^+ u^+, v^- \rangle + \langle \sigma_i^- u^-, v^+ \rangle = \langle u^+, \sigma_i^- v^- \rangle + \langle u^-, \sigma_i^+ v^+ \rangle = \langle u, \sigma_i v \rangle$. Thus σ_1 and σ_2 are self-adjoint.

Let $(c_1, c_2) \in \mathbb{R}^2 \setminus \{0\}$. Then $c_1\sigma_1^- + c_2\sigma_2^- = \sigma_1^-(c_1 + c_2 Q^-)$ is an isomorphism of E^- onto $(E^-)^\perp$. By definition of σ_i^+, $\langle (c_1\sigma_1^+ + c_2\sigma_2^+) u, v \rangle = \langle u, (c_1\sigma_1^- + c_2\sigma_2^-) v \rangle$ for any $u \in E_x^+$, $v \in E_x^-$. Therefore, $c_1\sigma_1^+ + c_2\sigma_2^+$ is an isomorphism of E^+ onto $(E^+)^\perp$. The direct sum decompositions $E^- \oplus E^+ = E = (E^-)^\perp \oplus (E^+)^\perp$ imply that $c_1\sigma_1 + c_2\sigma_2$ is a smooth bundle automorphism of E. Thus (σ_1, σ_2) determines the self-adjoint elliptic symbol $\sigma \in \Sigma(E)$, $\sigma(e_i) = \sigma_i$.

The automorphism $Q = \sigma_1^{-1}\sigma_2$ of E leaves E^- and E^+ invariant, and the restriction of Q to E^- coincides with Q^-. All eigenvalues of Q^- have negative imaginary part. Ranks of E^- and E^+ coincide, so by Proposition 15.1 all eigenvalues of the restriction of Q to E^+ have positive imaginary part.

By construction, $\vartheta(\sigma) = (E^-, E^+, J, Q^-)$. The same construction shows that σ is determined uniquely by the quadruple (E^-, E^+, J, Q^-). Therefore ϑ defines a bijection between $\Sigma(E)$ and $\Theta(E)$.

By Proposition 16.3, ϑ is continuous. The construction of the inverse map given above shows that ϑ^{-1} is also continuous. This completes the proof of the lemma. □

Continuation of the proof of Proposition 20.3. By this lemma, instead of a deformation retraction of $\Sigma(E)$ we can construct a deformation retraction of $\Theta(E)$ onto the subspace

$$\Theta^D(E) = \vartheta(\Sigma^D(E)) = \left\{(E^-, E^+, J, Q^-) \in \Theta(E)\colon\ E^+ = (E^-)^\perp,\ J \in U(E^-, E^+),\ Q^- = -i\,\mathrm{Id}\right\}.$$

For fixed E^-, all three ingredients of the triple (E^+, J, Q^-) can be deformed independently of one another. We define a homotopy $h_s(E^-, E^+, J, Q^-) = (E^-, E^+_s, J_s, Q^-_s)$ by the formulas

$$J_s = \left(s(JJ^*)^{-1/2} + 1 - s\right) J, \quad Q^-_s = -is\,\mathrm{Id} + (1-s)Q^-,$$

and E^+_s be the graph of $(1-s)B$, where B is the smooth homomorphism from $(E^-)^\perp$ to E^- with the graph E^+.

Obviously, $h_0 = \mathrm{Id}$, the image of h_1 is contained in $\Theta^D(E)$, and the restriction of h_s to $\Theta^D(E)$ is the identity for all $s \in [0,1]$. Thus h defines a deformation retraction of $\Sigma(E)$ onto $\Sigma^D(E)$. By construction, h_s is $U(E)$-equivariant and preserves $E^-(\sigma)$ for every $s \in [0,1]$. This completes the proof of the Proposition. □

Retraction of operators. Using results of Propositions 20.1–20.3, we are now able to prove the following result.

Proposition 20.5. *The natural embedding* $\mathrm{Dir}(E) \hookrightarrow \mathrm{Ell}(E)$ *is a homotopy equivalence. Moreover, there exists a deformation retraction* H *of* $\mathrm{Ell}(E)$ *onto a subspace of* $\mathrm{Dir}(E)$ *having the following properties for all* $s \in [0,1]$ *and* $A \in \mathrm{Ell}(E)$*, with* $A_s = H_s(A)$*:*

(1) $E^-(A_s) = E^-(A)$.

(2) The symbol of A_s *depends only on* s *and the symbol* σ_A *of* A.

(3) The map $H_s\colon \sigma_A \mapsto \sigma_{A_s}$ *defined by (2) is* $U(E)$*-equivariant.*

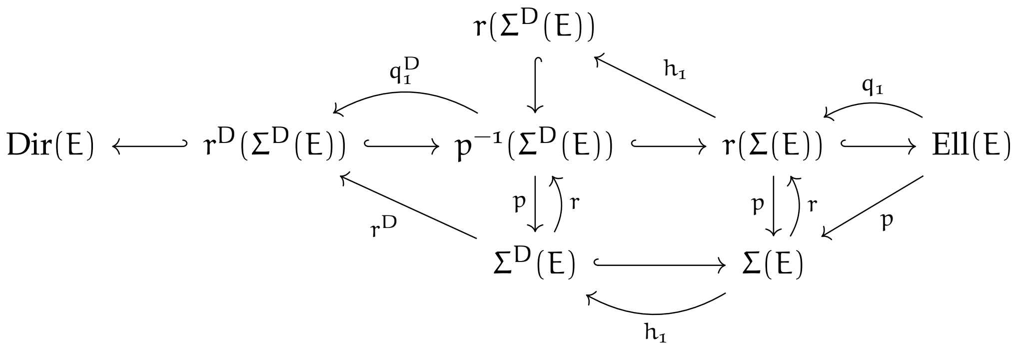
(4) If $A \in \mathrm{Dir}(E)$*, then* $\sigma_{A_s} = \sigma_A$.

(5) If $A, B \in \mathrm{Im}\, H_1$ *and the symbols of* A *and* B *coincide, then* $A = B$.

We will need only properties (1-3) in Part V. Properties (4-5) will be used below in Part VI.

Proof. Throughout the proof, we call a homotopy $[0,1] \times \mathrm{Ell}(E) \to \mathrm{Ell}(E)$ "nice" if it satisfies conditions (1-3) of the proposition. Obviously, the set of nice homotopies is closed under concatenation. We will construct a desired deformation retraction H as the concatenation of three nice homotopies. Then we show that the resulting homotopy satisfies conditions (4-5) as well.

Let $r\colon \Sigma(E) \to \mathrm{Ell}(E)$ be a section from Proposition 20.1 and $r^D\colon \Sigma^D(E) \to \mathrm{Dir}(E)$ be a section from Proposition 20.2. The linear fiberwise homotopy q between $r \circ p$ and the identity map is a nice deformation retraction of $\mathrm{Ell}(E)$ onto $r(\Sigma(E))$. The composition $r \circ h_s \circ p$ gives a nice deformation retraction of $r(\Sigma(E))$ onto $r(\Sigma^D(E)) \subset p^{-1}(\Sigma^D(E))$; we will denote it by the same letter h. The linear fiberwise homotopy q^D between $r^D \circ p$ and the identity map is a nice deformation retraction of $p^{-1}(\Sigma^D(E))$ onto $r^D(\Sigma^D(E)) \subset \mathrm{Dir}(E)$.



Concatenating q, h, and q^D, we obtain a nice deformation retraction H of $\mathrm{Ell}(E)$ onto the subspace $r^D(\Sigma^D(E))$ of $\mathrm{Dir}(E)$:

$$H_s(A) = \begin{cases} q_{3s}(A) \text{ for } 0 \leqslant s \leqslant 1/3, \\ h_{3s-1}q_1(A) \text{ for } 1/3 \leqslant s \leqslant 2/3 \\ q^D_{3s-2}h_1 q_1(A) \text{ for } 2/3 \leqslant s \leqslant 1. \end{cases}$$

If $A \in \mathrm{Dir}(E)$, then $\sigma_A \in \Sigma^D(E)$, so the symbol of A_s is independent of s. If $A \in \mathrm{Im}\, H_1$, then $A = r^D(\sigma_A)$. This proves conditions (4-5) of the proposition.

It remains to check that the natural embedding $\mathrm{Dir}(E) \hookrightarrow \mathrm{Ell}(E)$ is a homotopy equivalence. For every $A \in \mathrm{Dir}(E)$, the image $H_1(A) = A_1$ also lies in $\mathrm{Dir}(E)$, but we need to be careful because A_s is not necessarily odd for $s \in (0, 1)$. By property (4) the symbols of A_1 and A coincide. Thus the formula $H'_s(A) = (1-s)A + sH_1(A)$ defines a continuous map $H' \colon [0,1] \times \mathrm{Dir}(E) \to \mathrm{Dir}(E)$ such that $H'_0 = \mathrm{Id}$ and $H'_1 = H_1$. It follows that the restriction of H_1 to $\mathrm{Dir}(E)$ and the identity map $\mathrm{Id}_{\mathrm{Dir}(E)}$ are homotopic as maps from $\mathrm{Dir}(E)$ to $\mathrm{Dir}(E)$. On the other hand, the map $H_1 \colon \mathrm{Ell}(E) \to \mathrm{Ell}(E)$ is homotopic to $\mathrm{Id}_{\mathrm{Ell}(E)}$ via the homotopy H. It follows that $H_1 \colon \mathrm{Ell}(E) \to \mathrm{Dir}(E)$ is homotopy inverse to the embedding $\mathrm{Dir}(E) \hookrightarrow \mathrm{Ell}(E)$, that is, this embedding is a homotopy equivalence. This completes the proof of the proposition. □

Proposition 20.6. *The natural embedding* $\overline{\mathrm{Dir}}(E) \hookrightarrow \overline{\mathrm{Ell}}(E)$ *is a homotopy equivalence. Moreover, there exists a deformation retraction of* $\overline{\mathrm{Ell}}(E)$ *onto a subspace of* $\overline{\mathrm{Dir}}(E)$ *preserving both* $E^-(A)$ *and* $F(A, L)$.

Proof. Since the deformation retraction H constructed in Proposition 20.5 preserves $E^-(A)$, one can define the deformation retraction $\bar{H} \colon [0,1] \times \overline{\mathrm{Ell}}(E) \to \overline{\mathrm{Ell}}(E)$ covering H and satisfying the conditions of the proposition by the formula $\bar{H}_s(A, T) = (H_s(A), T)$ for $(A, T) \in \overline{\mathrm{Ell}}(E)$. □

Retraction of paths. Applying the deformation retraction from last two propositions point-wise and slightly correcting it on the ends of a path, we obtain a deformation retraction of the space of paths in $\mathrm{Ell}(E)$ and in $\overline{\mathrm{Ell}}(E)$.

Proposition 20.7. *Let* $g \in U(E)$*. Then the following two statements hold.*

1. *There exists a deformation retraction of* $\Omega_g \operatorname{Ell}(E)$ *onto a subspace of* $\Omega_g \operatorname{Dir}(E)$ *preserving* $\mathcal{E}^-(\gamma, g)$ *for every* $\gamma \in \Omega_g \operatorname{Ell}(E)$*.*

2. *There exists a deformation retraction of* $\Omega_g \overline{\operatorname{Ell}}(E)$ *onto a subspace of* $\Omega_g \overline{\operatorname{Dir}}(E)$ *preserving both* $\mathcal{E}^-(\gamma, g)$ *and* $\mathcal{F}(\gamma, g)$*.*

Proof. 1. Let $\rho_0, \rho_1 \colon [0,1] \to \mathbb{R}$ be a partition of unity subordinated to the covering $[0,1] = U_0 \cup U_1$, $U_0 = [0, 2/3)$, $U_1 = (1/3, 1]$, that is, $\operatorname{supp} \rho_i \subset U_i$ and $\rho_0 + \rho_1 \equiv 1$. Let $h \colon [0,1] \times \operatorname{Ell}(E) \to \operatorname{Ell}(E)$ be a deformation retraction of $\operatorname{Ell}(E)$ onto a subspace of $\operatorname{Dir}(E)$ satisfying the conditions of Proposition 20.5. Then a desired deformation retraction $[0,1] \times \Omega_g \operatorname{Ell}(E) \to \Omega_g \operatorname{Ell}(E)$ can be defined by the formula

$$(s, \mathcal{A}) \mapsto \mathcal{A}_s = \rho_0 \mathcal{A}_s^0 + \rho_1 \mathcal{A}_s^1, \text{ where } A_s^0(t) = h_s(\mathcal{A}(t)),\ \mathcal{A}_s^1(t) = g h_s(g^{-1}\mathcal{A}(t)). \tag{20.2}$$

Indeed, by property (3) of Proposition 20.5 the operators $\mathcal{A}_s^0(t)$ and $\mathcal{A}_s^1(t)$ have the same symbols, so their convex combination $A_s(t)$ lies in $\operatorname{Ell}(E)$ for every $t \in [0,1]$. The symbols and the chiral decompositions of the odd Dirac operators $\mathcal{A}_1^0(t)$ and $\mathcal{A}_1^1(t)$ coincide, so their convex combination $\mathcal{A}_1(t)$ lies in $\operatorname{Dir}(E)$. For $s = 0$ we get $\mathcal{A}_0^0 = \mathcal{A}_0^1 = \mathcal{A}$, so $\mathcal{A}_0 = (\rho_0 + \rho_1)\mathcal{A} = \mathcal{A}$. For each $s \in [0,1]$ we have

$$\mathcal{A}_s(1) = \mathcal{A}_s^1(1) = g h_s(g^{-1}\mathcal{A}(1)) = g h_s(\mathcal{A}(0)) = g\mathcal{A}_s^0(0) = g\mathcal{A}_s(0),$$

so $\mathcal{A}_s$ lies in $\Omega_g \operatorname{Ell}(E)$. Since $E^-(A)$ depends only on the symbol of A and is preserved by h_s, we get $\mathcal{E}^-(\mathcal{A}_s, g) = \mathcal{E}^-(\mathcal{A}, g)$ for every $s \in [0,1]$.

2. We define the deformation retraction $H \colon [0,1] \times \Omega_g \overline{\operatorname{Ell}}(E) \to \Omega_g \overline{\operatorname{Ell}}(E)$ by the formula $H_s(\gamma)(t) = (\mathcal{A}_s(t), T(t))$ for $\gamma \in \Omega_g \overline{\operatorname{Ell}}(E)$, where $\mathcal{A}$ is the projection of γ to $\Omega_g \operatorname{Ell}(E)$, $\gamma(t) = (\mathcal{A}(t), T(t))$, and $\mathcal{A}_s$ is defined by the formula (20.2). Since $\mathcal{E}^-(\mathcal{A}_s, g) = \mathcal{E}^-(\mathcal{A}, g)$, $H_s(\gamma)$ is correctly defined. □

Deformation retraction of special subspaces. Let $\overline{\operatorname{Ell}}^+(E)$, respectively $\overline{\operatorname{Ell}}^-(E)$ be the subspace of $\overline{\operatorname{Ell}}(E)$ consisting of all (A, L) with positive definite T, respectively negative definite T (see Proposition 15.3).

Proposition 20.8. *For every* $g \in U(E)$*, there exists a deformation retraction of* $\Omega_g \overline{\operatorname{Ell}}^+(E)$ *onto a subspace of* $\Omega_g \overline{\operatorname{Dir}}^+(E)$ *and a deformation retraction of* $\Omega_g \overline{\operatorname{Ell}}^-(E)$ *onto a subspace of* $\Omega_g \overline{\operatorname{Dir}}^-(E)$*.*

Proof. Let H be a deformation retraction of $\Omega_g \overline{\operatorname{Ell}}(E)$ onto a subspace of $\Omega_g \overline{\operatorname{Dir}}(E)$ satisfying conditions of Proposition 20.7.

For $\gamma \in \Omega_g \overline{\operatorname{Ell}}^+(E)$ and $\gamma_s = H_s(\gamma)$ we have $\mathcal{F}(\gamma_s) = \mathcal{F}(\gamma) = 0$, so by Proposition 17.2 $\gamma_s \in \Omega_g \overline{\operatorname{Ell}}^+(E)$ for every s. In particular, $\gamma_1 \in \Omega_g \overline{\operatorname{Ell}}^+(E) \cap \Omega_g \overline{\operatorname{Dir}}(E) = \Omega_g \overline{\operatorname{Dir}}^+(E)$.

For $\gamma \in \Omega_g \overline{\operatorname{Ell}}^-(E)$ and $\gamma_s = H_s(\gamma)$ we have $\mathcal{F}(\gamma_s) = \mathcal{F}(\gamma) = \mathcal{E}^-(\gamma) = \mathcal{E}^-(\gamma_s)$, so by Proposition 17.2 $\gamma_s \in \Omega_g \overline{\operatorname{Ell}}^-(E)$ for every s. In particular, $\gamma_1 \in \Omega_g \overline{\operatorname{Ell}}^-(E) \cap \Omega_g \overline{\operatorname{Dir}}(E) = \Omega_g \overline{\operatorname{Dir}}^-(E)$. □

21 Vanishing of the spectral flow

Invertible Dirac operators. We have no means to detect the invertibility of an arbitrary element of $\overline{\mathrm{Ell}}(E)$ by purely topological methods. However, there is a big class of *odd Dirac* operators which are necessarily invertible.

Proposition 21.1. *Let $A \in \mathrm{Dir}(E)$, that is, A is an odd Dirac operator. Let T be a positive definite automorphism of $E^-_\partial(A)$, and let L be the boundary condition for A defined by (15.2). Then A_L has no zero eigenvalues. The same is true for negative definite T. In other words, both $\overline{\mathrm{Dir}}^+(E)$ and $\overline{\mathrm{Dir}}^-(E)$ are subspaces of $\overline{\mathrm{Ell}}^0(E)$.*

This proposition explains why we distinguish odd Dirac operators. If T is definite, but A is not odd, then A_L no longer has to be invertible.

Proof. Let A be defined by formula (18.1). Denote the symbol of A^+ by σ^+. Let $u = (u^+, u^-)$ be a section of the vector bundle $E = E^+(A) \oplus E^-(A)$. If $u \in \mathrm{dom}(A_L)$, then the restriction of u to ∂M satisfies $i\sigma^+(n)u^+ = Tu^-$. Since A^+ and A^- are formally conjugate one to another, Green's formula gives

$$\int_{\partial M} \langle Tu^-, u^- \rangle \, dl = \int_{\partial M} \langle i\sigma^+(n)u^+, u^- \rangle \, dl = \int_M \left(\langle A^+u^+, u^- \rangle - \langle u^+, A^-u^- \rangle \right) ds,$$

where dl is the length element on ∂M and ds is the volume element on M.

Suppose now that $A_L u = 0$. Then $A^+u^+ = A^-u^- = 0$, so the last integral vanishes and we obtain $\int_{\partial M} \langle Tu^-, u^- \rangle \, dl = 0$. If T is positive definite or negative definite on ∂M, then the last equality implies vanishing of u^- on ∂M. This together with the boundary condition $i\sigma^+(n)u^+ = Tu^-$ implies vanishing of u^+ on ∂M. By the weak inner unique continuation property of Dirac operators [BW], we get $u \equiv 0$ on whole M. It follows that A_L has no zero eigenvalues, which completes the proof of the proposition. □

Vanishing of the spectral flow. Our next goal is to show that the spectral flow satisfies the first condition of Theorem 19.3.

Proposition 21.2. *Let γ be an element of $\Omega^*\overline{\mathrm{Ell}}_M$, $\Omega_U\overline{\mathrm{Ell}}^+_M$, or $\Omega_U\overline{\mathrm{Ell}}^-_M$. Then γ is connected by a path with an element of $\Omega_U\overline{\mathrm{Ell}}^0_M$, and hence $\mathsf{sf}(\gamma) = 0$.*

Proof. Suppose that $\gamma \in \Omega_g\overline{\mathrm{Ell}}^+(E)$ or $\Omega_g\overline{\mathrm{Ell}}^-(E)$, $g \in U(E)$. By Proposition 20.8, γ is connected by a path with an element γ_1 of $\Omega_g\overline{\mathrm{Dir}}^+(E)$ or $\Omega_g\overline{\mathrm{Dir}}^-(E)$ respectively. By Proposition 21.1 $\gamma_1 \in \Omega_g\overline{\mathrm{Ell}}^0(E)$.

Suppose that $\gamma \in \Omega^* \mathrm{Ell}(E)$, that is, $\gamma(t) \equiv (A, L)$. Since A_L is Fredholm, $A_L - \lambda$ is invertible for some $\lambda \in \mathbb{R}$. The path $\gamma_s(t) = (A - s\lambda, L)$ connects γ with the constant loop $\gamma_1 \in \Omega^* \mathrm{Ell}^0(E)$.

Since the spectral flow vanishes on paths in $\overline{\mathrm{Ell}}^0(E)$, $\mathsf{sf}(\gamma_1) = 0$. The homotopy invariance of the spectral flow implies $\mathsf{sf}(\gamma) = 0$, which completes the proof of the proposition. □

22 The spectral flow formula

Now we are ready to compute the spectral flow.

Theorem 22.1. *Let* $\gamma\colon [0,1] \to \overline{\mathrm{Ell}}(E)$ *be a continuous path such that* $\gamma(1) = g\gamma(0)$ *for some smooth unitary bundle automorphism* g *of* E*. Then* $\operatorname{sf}(\gamma) = \Psi(\gamma, g)$*.*

The proof consists of a sequence of lemmas.

Lemma 22.2. *There is an integer* $c = c_M$ *depending only on* M *such that*

$$\operatorname{sf}(\gamma) = c \cdot \Psi(\gamma, g) \tag{22.1}$$

for every $\gamma \in \Omega_g \overline{\mathrm{Ell}}(E)$*,* $g \in U(E)$*.*

Proof. The spectral flow defines the homomorphism $\operatorname{sf}\colon \Omega_U \overline{\mathrm{Ell}}_M \to \mathbb{Z}$, $(\gamma, g) \mapsto \operatorname{sf}(\gamma)$, which is constant on path connected components of $\Omega_U \overline{\mathrm{Ell}}_M$. By Proposition 21.2, the spectral flow vanishes on $\Omega^* \overline{\mathrm{Ell}}_M$, $\Omega_U \overline{\mathrm{Ell}}^+_M$, and $\Omega_U \overline{\mathrm{Ell}}^-_M$. Thus $\Phi = \operatorname{sf}$ and $\Lambda = \mathbb{Z}$ satisfy the first condition of Theorem 19.3. By Theorem 19.3 there is a $c \in \mathbb{Z}$ such that (22.1) holds for every $\gamma \in \Omega_g \overline{\mathrm{Ell}}(2k_M)$. Since every vector bundle over M is trivial, this completes the proof of the lemma. □

Lemma 22.3. *The value of* c *does not depend on the choice of a metric on* M*.*

Proof. Let h, h' be two metrics on M. The Hilbert spaces $L^2(M, h; E)$ and $L^2(M, h'; E)$ are isomorphic, with an isometry given by the formula $u \mapsto cu$, where c is the positive-valued function on M defined by the formula $c = \sqrt{\det(h')/\det(h)}$. This isometry induces the bijection between the spaces $\overline{\mathrm{Ell}}(M, h; E)$ and $\overline{\mathrm{Ell}}(M, h'; E)$ and leaves invariant the spectral flow of paths. On the other hand, such an isometry leaves invariant both the symbols of operators and local boundary conditions, so it leaves invariant $F(A, L)$. The conjugation by c also leaves invariant bundle automorphism g. Therefore, the aforementioned bijection $\overline{\mathrm{Ell}}(M, h; E) \to \overline{\mathrm{Ell}}(M, h'; E)$ does not affect $\mathcal{F}(\gamma, g)$. This implies that the factor c in (22.1) is the same for metrics h and h'. Since h and h' are arbitrary metrics, c does not depend on the choice of a metric. □

Lemma 22.4. *If* M *is diffeomorphic to the annulus, then* $c_M = c_{\mathrm{ann}} = 1$*.*

This was proven by the author in [P1, Theorem 4] (c_{ann} is denoted by c_2 there). The proof is based on the direct computation of the spectral flow for the Dirac operator on $M = S^1 \times [0, 1]$ with varying connection and fixed boundary condition. We include the proof to the thesis for complicity.

Proof. By Lemma 22.3, the value of c_{ann} does not depend on the geometry of an annulus, so we can choose such a geometry that computation become as simple as possible. Let us take the annulus $M = \{(r, \varphi) : 1 \leqslant r \leqslant 2\}$ in polar coordinates (r, φ) on the plane, with the metric $ds^2 = dr^2 + d\varphi^2$. We compute the spectral flow for the path of Dirac operators $(D + Q_t)$ acting on sections of the trivial vector bundle over M of rank 2, that is, on functions

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \; u^\pm\colon M \to \mathbb{C}.$$

We take the Dirac operators $D + Q_t$ with the boundary condition given by formula (15.2), with a fixed (independent of t) scalar automorphism T. Let

$$D = -i\begin{pmatrix} 0 & \partial_r - i\partial_\varphi \\ \partial_r + i\partial_\varphi & 0 \end{pmatrix}, \quad Q_t = \begin{pmatrix} 0 & it \\ -it & 0 \end{pmatrix}, \quad T = \begin{cases} +1 \text{ at } r = 1 \\ -1 \text{ at } r = 2 \end{cases}.$$

The ends of this path are conjugated by the scalar function $g\colon M \to \mathcal{U}(1)$, $g = e^{i\varphi}$.

We obtain the following system for an eigenvector u and an eigenvalue λ of $(D + Q_t, T)$:

$$\begin{cases} (-i\partial_r + \partial_\varphi - it)u^+ = \lambda u^- \\ (-i\partial_r - \partial_\varphi + it)u^- = \lambda u^+ \\ u^+ = iu^- \text{ at } r = 1, 2 \end{cases}. \tag{22.2}$$

All the eigenvectors of $(D + Q_t, T)$ are smooth functions, so we can seek them in the form

$$u^\pm(r, \varphi) = \sum_{k\in\mathbb{Z}} u_k^\pm(r)e^{ik\varphi}. \tag{22.3}$$

Substituting (22.3) to (22.2), we obtain the following system:

$$\begin{cases} \partial_r u_k^+ - (k-t)u_k^+ - i\lambda u_k^- = 0 \\ \partial_r u_k^- + (k-t)u_k^- - i\lambda u_k^+ = 0 \\ u_k^+ = iu_k^- \text{ at } r = 1, 2 \end{cases}.$$

Equivalently,

$$\begin{cases} \partial_r\left(u_k^+ + iu_k^-\right) = (k-t-\lambda)\left(u_k^+ - iu_k^-\right) \\ \partial_r\left(u_k^+ - iu_k^-\right) = (k-t+\lambda)\left(u_k^+ + iu_k^-\right) \\ u_k^+ - iu_k^- = 0 \text{ at } r = 1, 2 \end{cases}$$

and $\partial_r^2\left(u_k^+ - iu_k^-\right) = ((k-t)^2 - \lambda^2)\left(u_k^+ - iu_k^-\right)$. So we have the following cases:

- either $u_k^+ = u_k^- \equiv 0$,
- or $k - t + \lambda = 0$, $u_k^- = \text{const}$, $u_k^+ = iu_k^-$,
- or $(k-t)^2 - \lambda^2 = -(\pi l)^2$, $l \in \mathbb{Z}\setminus\{0\}$, $u_k^+ - iu_k^- = \text{const}\cdot\left(e^{\pi i l r} - e^{-\pi i l r}\right)$.

Thus, the set

$$\Lambda = \{(t,\lambda)\colon \lambda \text{ is an eigenvalue of } (D + Q_t, T)\} \subset \mathbb{R}^2$$

can be represented as the union $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{(t,\lambda)\colon \lambda - t \in \mathbb{Z}\}$ (with multiplicities 1 for all eigenvalues) and Λ_2 is a subset of $\{(t,\lambda)\colon |\lambda| \geqslant \pi\}$.

If $\lambda_j(t)$ are continuous functions from the interval $[0,1]$ to $\mathbb{R}$ such that $\lambda_i(t) \leqslant \lambda_j(t)$ for $i \leqslant j$ and $\Lambda \cap \{(t,\lambda)\colon 0 \leqslant t \leqslant 1\}$ is the union of the graphs of functions $\lambda_j(t)$, then $\lambda_j(t) = j + t$ when $-3 \leqslant j \leqslant 2$ (up to a shift of the numeration). So

$$\mathsf{sf}\,(D + Q_t, T)_{t\in[0,1]} = 1.$$

On the other hand, by Lemma 22.2

$$\mathsf{sf}\,(D+Q_t,T)_{t\in[0,1]} = c_{ann}\cdot\Psi((D+Q_t),g).$$

Since T is positive on the inner boundary circle $\partial M_1 = \{r=1\}$, the restriction $\mathcal{F}_1$ of $\mathcal{F}=\mathcal{F}((D+Q_t),g)$ to this boundary circle vanishes. The restriction $\mathcal{F}_2$ of $\mathcal{F}$ to the outer boundary circle $\partial M_2=\{r=2\}$ is obtained from the trivial vector bundle of rank 1 over $\partial M_2\times[0,1]$ by gluing it with twisting by g. Therefore,

$$\Psi((D+Q_t),g)=\sum_{j=1}^{2} c_1(\mathcal{F}_j)[\partial M_j\times S^1] = c_1(\mathcal{F}_2)[\partial M_2\times S^1] = \deg(\,g|_{\partial M_2}) = 1.$$

Taking all this together, we obtain $c_{ann}=1$. □

Lemma 22.5. *For any smooth oriented connected surface* M *the values of* c_M *and* c_{ann} *coincide.*

Proof. There are different ways to reduce the computation of c_M to the case of an annulus. Here we describe one of them, namely the splitting of M into two pieces: the smaller surface M' diffeomorphic to M and the collar M'' of the boundary. Following ideas of Kirk and Lesch from [KL], we take the Dirac operator which has the product form near boundary and choose mutually orthogonal boundary conditions on the sides of the cut. Then the spectral flow over M coincides with the sum of spectral flows over M' and M''. Since M'' is the disjoint union of annuli, this reasoning allows to reduce the computation of c_M to the computation for the annuli. Let us describe this procedure in more detail.

Let U be a collar neighbourhood of ∂M in M; we identify U with the product $(-2\varepsilon,0]\times\partial M$. Let (y,z) be the coordinates on U, with $y\in\partial M$, $z\in(-2\varepsilon,0]$, and (∂_z,∂_y) a positive oriented basis in TU. Equip M with a metric whose restriction to U has the product form $dl^2=dy^2+dz^2$.

Let $D\in \mathrm{Dir}(E)$ be an odd Dirac operator acting on sections of $E=E^+(D)\oplus E^-(D)$ with $E^+(D)=E^-(D)=2_M$. Adding a bundle automorphism to D if required, we can ensure that the restriction of D to U has the product form $D|_U=-i(\sigma_1\partial_z+\sigma_2\partial_y)$, where $\sigma_1=\begin{pmatrix}0 & \sigma_1^-\\ \sigma_1^+ & 0\end{pmatrix}$, $\sigma_2=\sigma_1 Q$, $Q=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$.

Let $\mathcal{F}$ be a vector bundle of rank 1 over $\partial M\times S^1$ such that $c_1(\mathcal{F})[\partial M\times S^1]\neq 0$. Choose the smooth embedding of $\mathcal{F}$ into the trivial vector bundle of rank 2 over $\partial M\times S^1$. Restricting this embedding to $\partial M\times\{t\}$, we obtain the smooth loop $(F_t)_{t\in S^1}$ of smooth subbundles F_t of $2_{\partial M}$. Define the smooth automorphisms T_t of $2_{\partial M}$ by the formula $T_t=(-1)_{F_t}\oplus 1_{F_t^\perp}$. Let $L_t\subset E_\partial$ be the corresponding boundary condition for D (that is, L_t is obtained from T_t as described in Proposition 15.3). Then $\mathcal{F}=\mathcal{F}(\gamma)$ for the loop $\gamma\in\Omega\overline{\mathrm{Ell}}(E)$ defined by the formula $\gamma(t)=(D,L_t)$. By Lemma 22.2,

$$\mathsf{sf}\,(D,L_t)=c_M\cdot c_1(\mathcal{F})[\partial M\times S^1].$$

Let us cut M along $N=\{-\varepsilon\}\times\partial M\subset U$. We obtain the disconnected surface $M^{cut}=M'\amalg M''$, where $M''=[-\varepsilon,0]\times\partial M$ is the disjoint union of annuli and

M' is diffeomorphic to M. Denote by $E^{cut} = E' \amalg E''$ the lifting of E on M^{cut}, and by $D^{cut} = D' \amalg D''$ the lifting of D on M^{cut}. By N', N'' denote the sides of the cut, so that $\partial M' = N'$ and $\partial M'' = N'' \amalg \partial M$.

The restriction of E^{cut} to $N' \amalg N''$ is isomorphic to the disjoint union of two copies of $E|_N$. Let us identify its sections with sections of the vector bundle $\bar{E}_\partial = (E \oplus E)|_N$. The diagonal subbundle $\Delta = \{u \oplus u\}$ of $\bar{E}_\partial$ defines the so called transmission boundary condition on the cut. The natural isometry $L^2(E) \to L^2(E^{cut})$ takes the operator D_{L_t} to the operator $D^{cut}_{\Delta \amalg L_t}$. Therefore, $D^{cut}_{\Delta \amalg L_t}$ is a self-adjoint Fredholm regular operator on $L^2(E^{cut})$, and

$$\mathrm{sf}\,(D, L_t) = \mathrm{sf}\,(D^{cut}, \Delta \amalg L_t).$$

Extending the identification above to the identification of sections of $E^{cut}|_{U' \amalg U''}$ with sections of $\bar{E} = (E \oplus E)|_{U'}$, where $U' = (-2\varepsilon, -\varepsilon] \times \partial M$, $U'' = [-\varepsilon, 0) \times \partial M$, we can write D^{cut} in the collar of the cut as

$$\bar{D} = -i(\bar{\sigma}_1 \partial_{\bar{z}} + \bar{\sigma}_2 \partial_y), \text{ where } \bar{\sigma}_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \ \bar{\sigma}_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

and $\bar{z}$ is the normal coordinate increasing in the direction of the cut (so $\bar{z} = z$ on U' and $\bar{z} = -z - 2\varepsilon$ on U''). We also change the orientation on M', so that $(\partial_{\bar{z}}, \partial_y)$ becomes a negative oriented basis. Then $\bar{Q} = -\bar{\sigma}_1^{-1}\bar{\sigma}_2 = (-Q) \oplus Q$ and

$$\bar{E}^+ = E'^- \oplus E''^+, \quad \bar{E}^- = E'^+ \oplus E''^-. \tag{22.4}$$

The restriction $\bar{\sigma}_1^+$ of $\bar{\sigma}_1$ to $\bar{E}^+$ has the form $\bar{\sigma}_1^+ = \sigma_1^- \oplus (-\sigma_1^+)$ with respect to decompositions (22.4).

Proposition 15.3 associates with every self-adjoint automorphism $\bar{T}$ of $\bar{E}_\partial^-$ the subbundle $\bar{L}(\bar{T})$ of $\bar{E}_\partial$ given by the formula $i\bar{\sigma}_1^+ \bar{u}^+ = \bar{T}\bar{u}^-$. Each $\bar{L}(\bar{T})$ is a self-adjoint well posed boundary condition for $\bar{D}$ on the cut, so $\bar{L}(\bar{T}) \amalg L_t$ is a self-adjoint well posed boundary condition for D^{cut}.

The transmission boundary condition Δ corresponds to the unitary self-adjoint automorphism

$$\bar{T}_\Delta = i\bar{\sigma}_1^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma_1^- \\ -i\sigma_1^+ & 0 \end{pmatrix}$$

of $\bar{E}_\partial^- = 4_{\partial M}$. Over every point $x \in \partial M$ the trace of $\bar{T}_\Delta$ is zero, so it has exactly two positive and two negative eigenvalues. $\bar{T}_\Delta$ can be identified with a map from ∂M to the complex Grassmanian $\mathrm{Gr}(2,4)$. Since $\mathrm{Gr}(2,4)$ is simply connected, every two maps from ∂M to $\mathrm{Gr}(2,4)$ are homotopic. Thus $\bar{T}_0 = \bar{T}_\Delta$ can be connected by a smooth homotopy $(\bar{T}_s)$ with $\bar{T}_1 = (-1) \oplus 1$ in the space of (unitary) self-adjoint bundle automorphisms of $\bar{E}_\partial^-$.

Denote by $\bar{L}_s$ the subbundle of $\bar{E}_\partial$ corresponding to $\bar{T}_s$, and let $\bar{L} = \bar{L}_1$. Then $\bar{L}_s \amalg L_t$ is a self-adjoint well posed global boundary condition for D^{cut}, so $D^{cut}_{\bar{L}_s \amalg L_t}$ is a regular self-adjoint Fredholm operator on $L^2(E^{cut})$ for each s, t. By Lemma 13.4 from Part III, the map

$$[0,1] \times S^1 \to \mathrm{Gr}\left(H^{1/2}(\bar{E}_\partial) \oplus H^{1/2}(E_\partial)\right) \cong \mathrm{Gr}\left(H^{1/2}(E_\partial^{cut})\right),$$

$(s,t) \mapsto H^{1/2}(\bar{L}_s) \oplus H^{1/2}(L_t)$, is continuous. By Proposition 13.2, this implies the continuity of the map

$$[0,1] \times S^1 \to \mathcal{R}^{sa}_F(L^2(E^{cut})), \quad (s,t) \mapsto D^{cut}_{\bar{L}_s \amalg L_t}.$$

Therefore, by the homotopy invariance property of the spectral flow we have

$$\mathsf{sf}\,(D^{cut}, \Delta \amalg L_t) = \mathsf{sf}\,(D^{cut}, \bar{L} \amalg L_t).$$

The boundary condition $\bar{L}$ is given by the formula $i\bar{\sigma}_1^+ \bar{u}^+ = \bar{T}_1 \bar{u}^-$. Coming back from $\bar{E}_\partial$ to $E^{cut}|_{N' \amalg N''}$, we obtain $L' \amalg L''$ in place of $\bar{L}$, where L' is the subbundle of $E^{cut}|_{N'}$ given by the formula $i(-\sigma_1^+)u'^+ = u'^-$ and L'' is the subbundle of $E^{cut}|_{N''}$ given by the formula $i\sigma_1^+ u''^+ = u''^-$. Therefore, $\bar{L} \amalg L_t$ is a *local* boundary condition for D^{cut}. Applying Lemma 22.2 to the connected components of M^{cut}, we obtain

$$\begin{aligned}\mathsf{sf}\,(D^{cut}, \bar{L} \amalg L_t) &= \mathsf{sf}\,(D', L') + \mathsf{sf}\,(D'', L'' \amalg L_t) = \mathsf{sf}\,(D'', L'' \amalg L_t) = \\ &c_{ann} \cdot \big(c_1(\mathcal{F}'')[N'' \times S^1] + c_1(\mathcal{F})[\partial M \times S^1]\big) = c_{ann} \cdot c_1(\mathcal{F})[\partial M \times S^1],\end{aligned}$$

since $\mathcal{F}''$ is zero vector bundle.

Combining all this together, we obtain

$$c_M \cdot c_1(\mathcal{F})[\partial M \times S^1] = \mathsf{sf}\,(D, L_t) = \mathsf{sf}\,(D^{cut}, L' \amalg L'' \amalg L_t) = c_{ann} \cdot c_1(\mathcal{F})[\partial M \times S^1].$$

The value of $c_1(\mathcal{F})[\partial M \times S^1]$ does not vanish due to the choice of $\mathcal{F}$. Therefore, $c_M = c_{ann}$, which completes the proof of the lemma and of Theorem 22.1. □

23 Example: conjugation by a scalar function

In this section we illustrate Theorem 22.1 by a simple example. We consider a one-parameter family (A_t) of operators such that A_1 is conjugate to A_0 by a *scalar* automorphism g. We suppose that all A_t have the same symbol, and we take the same boundary condition for all A_t. In this case, our spectral flow formula takes an especially simple form.

Let $(A, L) \in \overline{\mathcal{E}ll}(E)$. Suppose that

$$g \colon M \to U(1) = \{z \in \mathbb{C} \colon |z| = 1\}$$

is a smooth function (scalar gauge transformation). Then the conjugation by g preserves the symbol of A. More precisely, the conjugation by g takes A to the operator $gAg^{-1} = A - g^{-1}\sigma_A(dg)$.

Let $Q \colon [0,1] \to \operatorname{End}^{sa}(E)$ be a one-parameter family of smooth self-adjoint bundle endomorphisms such that $Q_1 = Q_0 - g^{-1}\sigma_A(dg)$. Then $A_t = A + Q_t$ is a one-parameter family of self-adjoint elliptic operators satisfying the conjugation condition $A_1 = gA_0g^{-1}$.

Let T be the bundle automorphism of E^-_∂ defined by formula (15.4). Since $T(x)$ is nondegenerate at every $x \in \partial M$, the number of negative eigenvalues of $T(x)$ (counting multiplicities) does not depend on the choice of $x \in \partial M_j$. Denote this number by ε_j.

Theorem 23.1. *Let* $(A, L) \in \overline{\mathrm{Ell}}(E)$ *and let* $g\colon M \to \{z \in \mathbb{C}\colon |z| = 1\}$ *be a scalar gauge transformation. Suppose that* $Q\colon [0,1] \to \mathrm{End}^{sa}(E)$ *is a one-parameter family of smooth self-adjoint bundle endomorphisms such that* $Q_1 = Q_0 - g^{-1}\sigma_A(dg)$*. Then the spectral flow of the family* $(A + Q_t, L)$ *is given by the formula*

$$\mathsf{sf}\,(A + Q_t, L) = \sum_{j=1}^{m} \varepsilon_j g_j,$$

where g_j *is the degree of the restriction of* g *to* ∂M_j *and* ε_j *is defined as described above.*

Proof. Every local boundary condition is invariant with respect to a scalar bundle automorphisms. Thus the one-parameter family of boundary value problems $\gamma\colon [0,1] \to \overline{\mathrm{Ell}}(E)$ defined by the formula $\gamma(t) = (A + Q_t, L)$ is an element of $\Omega_g\overline{\mathrm{Ell}}(E)$.

Let us lift $F(A, L)$ to $\partial M \times [0,1]$ and identify $(u, 1)$ with $(gu, 0)$ for every $u \in F(A, L)$. The resulting vector bundle $\mathcal{F}$ over $\partial M \times S^1$ is isomorphic to $\mathcal{F}(\gamma, g)$. By Theorem 22.1,

$$\mathsf{sf}\,(\gamma) = c_1(\mathcal{F})[\partial M \times S^1] = \sum_{j=1}^{m} c_1(\mathcal{F}_j)[\partial M_j \times S^1].$$

The j-th summand in the right hand side is equal to the rank ε_j of $\mathcal{F}_j$ multiplied by the "twisting" number g_j. It follows that $\mathsf{sf}\,(A + Q_t, L) = \sum \varepsilon_j g_j$. □

24 Universality of the spectral flow

The direct sum of two invertible operators is again invertible, so the disjoint union

$$\overline{\mathrm{Ell}}^0_M = \coprod_{k \in \mathbb{N}} \overline{\mathrm{Ell}}^0(2k_M)$$

is a subsemigroup of $\overline{\mathrm{Ell}}_M$. The disjoint union

$$\Omega_U\overline{\mathrm{Ell}}^0_M = \coprod_{k \in \mathbb{N},\, g \in U(2k_M)} \Omega_g\overline{\mathrm{Ell}}^0(2k_M).$$

is a subsemigroup of $\Omega_U\overline{\mathrm{Ell}}_M$.

Theorem 24.1. *Let* Φ *be a semigroup homomorphism from* $\Omega_U\overline{\mathrm{Ell}}_M$ *to a commutative monoid* Λ*, which is constant on path connected components of* $\Omega_U\overline{\mathrm{Ell}}_M$*. Then the following two conditions are equivalent:*

1. Φ *vanishes on* $\Omega_U\overline{\mathrm{Ell}}^0_M$*.*
2. $\Phi = \vartheta \circ \mathsf{sf}$ *for some (unique) monoid homomorphism* $\vartheta\colon \mathbb{Z} \to \Lambda$*, that is,* Φ *has the form* $\Phi(\gamma, g) = c \cdot \mathsf{sf}(\gamma)$ *for some invertible constant* $c \in \Lambda$*.*

In other words, the spectral flow defines an isomorphism of monoids

$$\mathsf{sf}\colon \pi_0(\Omega_U \overline{\mathrm{Ell}}_M)/\pi_0(\Omega_U \overline{\mathrm{Ell}}_M^0) \to \mathbb{Z}.$$

Proof. (2 $\Rightarrow$ 1) follows immediately from properties (S0-S2) of the spectral flow; see Section 7.

Let us prove (1 $\Rightarrow$ 2). Suppose that Φ satisfies condition (1) of the theorem. By Proposition 21.2 Φ vanishes on $\Omega^* \overline{\mathrm{Ell}}_M$, $\Omega_U \overline{\mathrm{Ell}}_M^+$, and $\Omega_U \overline{\mathrm{Ell}}_M^-$. Theorem 19.3 then implies that $\Phi = \vartheta \circ \Psi$ for some (unique) monoid homomorphism $\vartheta\colon \mathbb{Z} \to \Lambda$. By Theorem 22.1 Ψ is equal to the spectral flow. Taking this all together, we obtain $\Phi = \vartheta \circ \mathsf{sf}$. Taking $c = \vartheta(1)$, we obtain $\Phi(\gamma, g) = c \cdot \mathsf{sf}(\gamma)$, which completes the proof of the theorem. $\square$

Part VI

Family index

25 The analytical index

The analytical index of a map. Recall that we have the natural inclusion

$$\iota\colon \overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_{K}\left(L^2(E)\right), \quad (A, L) \mapsto A_L,$$

and by Proposition 14.3 this inclusion is continuous.

Let γ be a continuous map from a compact topological space X to $\overline{\mathrm{Ell}}(E)$. We define ***the analytical index*** of γ to be the index of the composition of γ with the inclusion $\iota\colon \overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_{K}(L^2(E))$ and will denote it by $\mathrm{ind}_a(\gamma)$.

More generally, the index can be defined for a family of elliptic operators acting on a family of bundles; we describe such a situation below.

Families of elliptic operators. For a smooth Hermitian vector bundle E over M, we denote by $U(E)$ the group of smooth unitary bundle automorphisms of E with the C^1-topology.

The continuous action of the topological group $U(E)$ on E induces the continuous embedding $U(E) \hookrightarrow \mathcal{U}(L^2(E))$. The action of $U(E)$ on $\overline{\mathrm{Ell}}(E)$ given by the rule $g(A, L) = (gAg^{-1},\ gL)$ is continuous and compatible with the action of $U(E)$ on $\mathcal{R}(L^2(E))$.

Denote by Vect_X the class of all Hermitian vector bundles over X and by $\mathrm{Vect}^{\infty}_{M}$ the class of all smooth Hermitian vector bundles over M. Denote by $\mathrm{Vect}_{X,M}$ the class of all locally trivial fiber bundles over X with fibers $\mathcal{E}_x \in \mathrm{Vect}^{\infty}_{M}$ and the structure group $U(\mathcal{E}_x)$. Note that in the case of disconnected X the fibers over different points of X are not necessarily isomorphic.

Let $\mathcal{E} \in \mathrm{Vect}_{X,M}$. We will denote by $\overline{\mathrm{Ell}}(\mathcal{E})$ the locally trivial fiber bundle over X with the fiber $\overline{\mathrm{Ell}}(\mathcal{E}_x)$ associated with $\mathcal{E}$. A section of $\overline{\mathrm{Ell}}(\mathcal{E})$ is just a family of elliptic operators acting on fibers of a family $(\mathcal{E}_x)$ of vector bundles over M parametrized by points of X. We denote by $\Gamma\overline{\mathrm{Ell}}(\mathcal{E})$ the space of sections of $\overline{\mathrm{Ell}}(\mathcal{E})$ equipped with the compact-open topology.

The analytical index of a family. A bundle $\mathcal{E} \in \mathrm{Vect}_{X,M}$ defines the Hilbert bundle $\mathcal{H} = \mathcal{H}(\mathcal{E})$ over X, whose fiber over $x \in X$ is $\mathcal{H}_x = L^2(\mathcal{E}_x)$. Note that the fibers $\mathcal{H}_x$ over different points x are isomorphic as Hilbert spaces even if $\mathcal{E}_x$ are not isomorphic as vector bundles over M.

The natural embedding $\iota\colon \overline{\mathrm{Ell}}(E) \hookrightarrow \mathcal{R}^{sa}_{K}(L^2(E))$ is $U(E)$-equivariant and thus defines the bundle embedding $\overline{\mathrm{Ell}}(\mathcal{E}) \hookrightarrow \mathcal{R}^{sa}_{K}(\mathcal{H})$, which we still will denote by ι. For a section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$, $\iota(\gamma)$ is a section of $\mathcal{R}^{sa}_{K}(\mathcal{H})$. The *analytical index* $\mathrm{ind}_a(\gamma)$ of γ is defined as the family index of $\iota(\gamma)$.

Invertible operators. Recall that $\overline{\mathrm{Ell}}^0(E)$ denotes the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that the unbounded operator A_L has no zero eigenvalues (since A_L is self-adjoint, this condition is equivalent to the invertibility of A_L). For

$\mathcal{E} \in \mathrm{Vect}_{X,M}$ we denote by $\overline{\mathrm{Ell}}^o(\mathcal{E})$ the subbundle of $\overline{\mathrm{Ell}}(\mathcal{E})$, whose fiber over $x \in X$ is $\overline{\mathrm{Ell}}^o(\mathcal{E}_x)$.

Property (Io) of Proposition 8.2 implies that the analytical index vanishes on sections of $\overline{\mathrm{Ell}}^o(\mathcal{E})$; our proof of the index theorem will rely heavily upon this fact.

26 The topological index

The first main result of this part is the computation of the analytical index of a section $\gamma\colon x \mapsto (A_x, L_x)$ of $\overline{\mathrm{Ell}}(\mathcal{E})$ in terms of topological data of γ over the boundary. These data are encoded in the family $\mathcal{F} = (\mathcal{F}_x)_{x\in X}$ of vector bundles over ∂M with $\mathcal{F}_x = F(A_x, L_x)$.

The correspondence between boundary conditions and automorphisms of E_∂^-. Recall that by Proposition 15.3 self-adjoint elliptic local boundary conditions L for A are in a one-to-one correspondence with self-adjoint bundle automorphisms T of E_∂^-. This correspondence is given by the rule (15.4),

$$L = \operatorname{Ker} P_T \quad \text{with} \quad P_T = P^+ \left(1 + i\sigma(n)^{-1} T P^-\right),$$

where P^+ denotes the projection of E_∂ onto E_∂^+ along E_∂^- and $P^- = 1 - P^+$. It is shown in Proposition 16.4 that the correspondence $(A, L) \mapsto (A, T)$ is a homeomorphism. This allows us to move freely from (A, L) to (A, T) and back; we will use it further without special mention in constructions of homotopies.

In Section 15 we associated with a pair $(A, L) \in \overline{\mathrm{Ell}}(E)$ the subbundle $F = F(A, L)$ of E_∂^-, whose fibers F_x, $x \in \partial M$ are spanned by the generalized eigenspaces of T_x corresponding to negative eigenvalues. Being a subbundle of E_∂^-, $F(A, L)$ is also a smooth subbundle of E_∂. Moreover, the map $F\colon \overline{\mathrm{Ell}}(E) \to C^1(\mathrm{Gr}(E_\partial))$ is continuous; see Proposition 16.4.

Subbundles, restrictions, and forgetting smooth structure. For $\mathcal{V} \in \mathrm{Vect}_{X,M}$ we denote by $\mathcal{V}_\partial \in \mathrm{Vect}_{X,\partial M}$ the locally trivial bundle over X whose fiber over $x \in X$ is the restriction of $\mathcal{V}_x$ to ∂M.

Let N be a smooth manifold (in our case it will be either M or ∂M), and let $\mathcal{V} \in \mathrm{Vect}_{X,N}$. We say that $\mathcal{W} \subset \mathcal{V}$ is a *subbundle* of $\mathcal{V}$ if $\mathcal{W} \in \mathrm{Vect}_{X,N}$ and $\mathcal{W}_x$ is a smooth subbundle of $\mathcal{V}_x$ for every $x \in X$.

We will denote by $\langle \mathcal{V} \rangle$ the vector bundle over $X \times N$ whose restriction to $\{x\} \times N$ is the fiber $\mathcal{V}_x$ with forgotten smooth structure.

We discuss the correspondence $\mathcal{V} \mapsto \langle \mathcal{V} \rangle$ in more detail in the appendix to Part VI; we prove there a couple of technical results that are used in the main part of the thesis.

Definition of $\mathcal{F}(\gamma)$. Let γ be a section of $\overline{\mathrm{Ell}}(\mathcal{E})$, $\mathcal{E} \in \mathrm{Vect}_{X,M}$. By Propositions 16.3 and 16.4, $E^-(A_x)$ and $F(A_x, L_x) \subset E^-(A_x)$ continuously depend on x. Hence they define the subbundle $\mathcal{E}^-(\gamma)$ of $\mathcal{E}$ whose fiber over x is $E^-(A_x)$, and the subbundle $\mathcal{F} = \mathcal{F}(\gamma)$ of $\mathcal{E}^-_\partial(\gamma)$ whose fiber over x is $F(A_x, L_x)$.

The homomorphism Ind_t**.** The boundary ∂M is a disjoint union of circles, so the natural homomorphism

$$K^0(X) \otimes K^0(\partial M) \oplus K^1(X) \otimes K^1(\partial M) \longrightarrow K^0(X \times \partial M)$$

is an isomorphism. Denote by α_∂ the projection of $K^0(X \times \partial M)$ on the second summand $K^1(X) \otimes K^1(\partial M)$ of this direct sum. The orientation of M induces an orientation of ∂M and thus defines the identification of $K^1(\partial M) = \bigoplus_{j=1}^m K^1(\partial M_j)$ with $\mathbb{Z}^m$, where ∂M_j, $j = 1 \ldots m$, are the boundary components. Denote by δ the homomorphism $K^1(\partial M) = \mathbb{Z}^m \to \mathbb{Z}$ given by the formula $(a_1, \ldots, a_m) \mapsto \sum_{j=1}^m a_j$. Equivalently, δ is the connecting homomorphism of the exact sequence

$$K^1(M) \xrightarrow{i^*} K^1(\partial M) \xrightarrow{\delta} K^0(M, \partial M) = \mathbb{Z},$$

where i denotes the inclusion $\partial M \hookrightarrow M$ and the identification of $K^0(M, \partial M)$ with $\mathbb{Z}$ is given by the orientation of M.

We define the topological index homomorphism

$$\operatorname{Ind}_t\colon K^0(X \times \partial M) \to K^1(X)$$

to be the composition

$$K^0(X \times \partial M) \xrightarrow{\alpha_\partial} K^1(X) \otimes K^1(\partial M) \xrightarrow{\operatorname{Id} \otimes \delta} K^1(X) \otimes \mathbb{Z} = K^1(X). \tag{26.1}$$

The topological index. We define the topological index of a section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ by the formula

$$\operatorname{ind}_t(\gamma) = \operatorname{Ind}_t[\mathcal{F}(\gamma)], \tag{26.2}$$

where $[\mathcal{F}]$ denotes the class of $\langle \mathcal{F} \rangle$ in $K^0(X \times \partial M)$.

27 Properties of the topological index

Properties of the homomorphism Ind_t**.** Denote by G^∂ the image of the homomorphism $K^0(X \times M) \to K^0(X \times \partial M)$ induced by the embedding of $X \times \partial M$ in $X \times M$.

Denote by $G^\boxtimes$ the image of the natural homomorphism $K^0(X) \otimes K^0(\partial M) \to K^0(X \times \partial M)$. Recall that this homomorphism takes the tensor product $[W] \otimes [V]$ of the classes of vector bundles W over X and V over ∂M to the class of their external tensor product $[W \boxtimes V] \in K^0(X \times \partial M)$.

Denote by G the subgroup of $K^0(X \times \partial M)$ spanned by G^∂ and $G^\boxtimes$.

Proposition 27.1. *The homomorphism* Ind_t *is surjective with the kernel* G*. In other words, the following sequence is exact:*

$$0 \longrightarrow G \longrightarrow K^0(X \times \partial M) \xrightarrow{\operatorname{Ind}_t} K^1(X) \longrightarrow 0.$$

Proof. The groups $K^*(M)$ and $K^*(\partial M)$ are free of torsion, so the first two rows of the following commutative diagram are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^0(X) \otimes K^0(M) & \longrightarrow & K^0(X \times M) & \xrightarrow{\alpha} & K^1(X) \otimes K^1(M) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \operatorname{Id} \otimes i^*} & & \big\downarrow{\scriptstyle (\operatorname{Id} \times i)^*} & & \big\downarrow{\scriptstyle \operatorname{Id} \otimes i^*} & & \\
0 & \longrightarrow & K^0(X) \otimes K^0(\partial M) & \longrightarrow & K^0(X \times \partial M) & \xrightarrow{\alpha_\partial} & K^1(X) \otimes K^1(\partial M) & \longrightarrow & 0 \\
 & & & & & \searrow{\scriptstyle \operatorname{Ind}_t} & \big\downarrow{\scriptstyle \operatorname{Id} \otimes \delta} & & \\
 & & & & & & K^1(X) \otimes \mathbb{Z} & & \\
 & & & & & & \big\downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Taking tensor product of the exact sequence

$$K^1(M) \xrightarrow{i^*} K^1(\partial M) \xrightarrow{\delta} K^0(M, \partial M) = \mathbb{Z} \longrightarrow 0$$

by $K^1(X)$, we see that the right column of this diagram is also exact.

It follows from the diagram that Ind_t vanishes on both $G^{\boxtimes}$ and G^{∂}. Both α_∂ and $\operatorname{Id} \otimes \delta$ are surjective, so Ind_t is also surjective. Finally,

$$\begin{aligned} K^0(X \times \partial M)/G = \operatorname{Im}(\alpha_\partial) \,/\, \operatorname{Im}(\alpha_\partial \circ (\operatorname{Id} \times i)^*) = \\ = (K^1(X) \otimes K^1(\partial M)) \,/\, (K^1(X) \otimes K^1(M)) = K^1(X) \otimes \mathbb{Z}, \end{aligned}$$

and the quotient map is given by the composition $(\operatorname{Id} \otimes \delta) \circ \alpha_\partial = \operatorname{Ind}_t$. This completes the proof of the proposition. □

Special subspaces. The following two subspaces of $\overline{\mathrm{Ell}}(E)$ will play a special role:

- $\overline{\mathrm{Ell}}^+(E)$ consists of all $(A, T) \in \overline{\mathrm{Ell}}(E)$ with positive definite T.
- $\overline{\mathrm{Ell}}^-(E)$ consists of all $(A, T) \in \overline{\mathrm{Ell}}(E)$ with negative definite T.

Proposition 27.2. *Let* γ *be a section of* $\overline{\mathrm{Ell}}(\mathcal{E})$*. Then the following statements hold:*

- $\mathcal{F}(\gamma) = 0$ *if and only if* γ *is a section of* $\overline{\mathrm{Ell}}^+(\mathcal{E})$*;*
- $\mathcal{F}(\gamma) = \mathcal{E}^-_\partial(\gamma)$ *if and only if* γ *is a section of* $\overline{\mathrm{Ell}}^-(\mathcal{E})$*.*

Proof. This follows immediately from the definition of $\mathcal{F}$. □

Denote by $\Gamma^\pm\,\overline{\mathrm{Ell}}(\mathcal{E})$ the subspace of $\Gamma\overline{\mathrm{Ell}}(\mathcal{E})$ consisting of sections γ that can be written in the form

$$\gamma = \gamma' \oplus \gamma'' \;\text{ with }\; \gamma' \in \Gamma\overline{\mathrm{Ell}}^+(\mathcal{E}') \text{ and } \gamma'' \in \Gamma\overline{\mathrm{Ell}}^-(\mathcal{E}'') \qquad (27.1)$$

for some orthogonal decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}''$.

Proposition 27.3. *The class of* $\mathcal{F}(\gamma)$ *in* $K^0(X \times \partial M)$ *lies in* G^∂ *for every* $\gamma \in \Gamma^\pm \overline{\mathrm{Ell}}(\mathcal{E})$.

Proof. For γ defined by (27.1), $\mathcal{F}(\gamma) = \mathcal{F}(\gamma'') = \mathcal{E}^-_\partial(\gamma'')$, so $[\mathcal{F}(\gamma)] \in G^\partial$. □

Twisting. A bundle $\mathcal{E} \in \mathrm{Vect}_{X,M}$ can be twisted by $W \in \mathrm{Vect}_X$, giving rise to another bundle from $\mathrm{Vect}_{X,M}$, which we denote by $W \otimes \mathcal{E}$. If W is a subbundle of a trivial vector bundle k_X, then $W \otimes \mathcal{E}$ is a subbundle of the direct sum of k copies of $\mathcal{E}$, whose fiber over $x \in X$ is $W_x \otimes \mathcal{E}_x$.

A section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ can be twisted by W, resulting in the section $1_W \otimes \gamma$ of $\overline{\mathrm{Ell}}(W \otimes \mathcal{E})$. This construction induces the map $1_W \otimes \colon \overline{\mathrm{Ell}}(\mathcal{E}) \to \overline{\mathrm{Ell}}(W \otimes \mathcal{E})$.

For $W \in \mathrm{Vect}_X$ and $E \in \mathrm{Vect}^\infty_M$ we denote by $W \boxtimes E$ the tensor product $W \otimes \mathcal{E}$, where $\mathcal{E}$ is the trivial bundle over X with the fiber E. For $(A, L) \in \overline{\mathrm{Ell}}(E)$ we denote by $1_W \boxtimes (A, L)$ the section $1_W \otimes \gamma$ of $W \boxtimes E$, where $\gamma \colon X \to \overline{\mathrm{Ell}}(E)$ is the constant map $x \mapsto (A, L)$.

Denote by $\Gamma^\boxtimes \overline{\mathrm{Ell}}(\mathcal{E})$ the subspace of $\Gamma\overline{\mathrm{Ell}}(\mathcal{E})$ consisting of sections γ having the form

$$\gamma = \bigoplus_i 1_{W_i} \boxtimes (A_i, L_i) \tag{27.2}$$

for some $(A_i, L_i) \in \overline{\mathrm{Ell}}(E_i)$, $E_i \in \mathrm{Vect}^\infty_M$, and $W_i \in \mathrm{Vect}_X$ with respect to some decomposition of $\mathcal{E}$ into the orthogonal direct sum $\bigoplus_i W_i \boxtimes E_i$.

Proposition 27.4. *The class of* $\mathcal{F}(\gamma)$ *in* $K^0(X \times \partial M)$ *lies in* $G^\boxtimes$ *for every* $\gamma \in \Gamma^\boxtimes \overline{\mathrm{Ell}}(\mathcal{E})$.

Proof. For γ defined by formula (27.2) we have $[\mathcal{F}(\gamma)] = \sum_i [W_i \boxtimes F(A_i, L_i)] \in G^\boxtimes$. □

Properties of the topological index. A continuous map $f \colon X \to Y$ induces the map $f^*_\mathcal{E} \colon \Gamma\overline{\mathrm{Ell}}(\mathcal{E}) \to \Gamma\overline{\mathrm{Ell}}(f^*\mathcal{E})$ for every $\mathcal{E} \in \mathrm{Vect}_{Y,M}$. On the other hand, f induces the homomorphism $f^* \colon K^1(Y) \to K^1(X)$. We will use this functoriality to state property (T3) in the following proposition.

Proposition 27.5. *The topological index has the following properties for every* $\mathcal{E}, \mathcal{E}' \in \mathrm{Vect}_{X,M}$*:*

(T0) The topological index vanishes on $\Gamma^\pm$ Ell$(\mathcal{E})$ *and* $\Gamma^\boxtimes \overline{\mathrm{Ell}}(\mathcal{E})$.

(T1) $\mathrm{ind}_t(\gamma) = \mathrm{ind}_t(\gamma')$ *if* γ *and* γ' *are homotopic sections of* $\overline{\mathrm{Ell}}(\mathcal{E})$.

(T2) $\mathrm{ind}_t(\gamma \oplus \gamma') = \mathrm{ind}_t(\gamma) + \mathrm{ind}_t(\gamma') \in K^0(X)$ *for every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *and* γ' *of* $\overline{\mathrm{Ell}}(\mathcal{E}')$.

(T3) $\mathrm{ind}_t(f^*\gamma) = f^* \mathrm{ind}_t(\gamma) \in K^1(Y)$ *for any section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *and any continuous map* $f \colon Y \to X$.

(T4) $\mathrm{ind}_t(1_W \otimes \gamma) = [W] \cdot \mathrm{ind}_t(\gamma)$ *for every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *and every* $W \in \mathrm{Vect}_X$.

(T5) For a loop $\gamma \colon S^1 \to \overline{\mathrm{Ell}}(E)$,

$$\mathrm{ind}_t(\gamma) = c_1(\mathcal{F}(\gamma))[\partial M \times S^1] \tag{27.3}$$

up to the natural identification $K^1(S^1) \cong \mathbb{Z}$. Here $c_1(\mathcal{F})$ is the first Chern class of $\mathcal{F}$, $[\partial M \times S^1]$ is the fundamental class of $\partial M \times S^1$, and ∂M is equipped with an orientation in such a way that the pair (outward normal to ∂M, positive tangent vector to ∂M) has a positive orientation.

Vanishing of ind_t on $\Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}(\mathcal{E})$ is a corollary of (T3) and (T4); however, we prefer to give this property separately in (T0) for a reason which will be clear later.

Proof. (T0). If $\gamma \in \Gamma^{\pm}\,\overline{\mathrm{Ell}}(\mathcal{E})$, then $[\mathcal{F}(\gamma)] \in G^{\partial}$ by Proposition 27.3. If $\gamma \in \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}(\mathcal{E})$, then $[\mathcal{F}(\gamma)] \in G^{\boxtimes}$ by Proposition 27.4. In both cases Proposition 27.1 implies $\operatorname{ind}_t(\gamma) = 0$.

(T1). If γ and γ' are homotopic sections of $\overline{\mathrm{Ell}}(\mathcal{E})$, then $\mathcal{F}(\gamma)$ and $\mathcal{F}(\gamma')$ are homotopic subbundles of $\mathcal{E}_\partial$. Thus the subbundles $\langle\mathcal{F}(\gamma)\rangle$ and $\langle\mathcal{F}(\gamma')\rangle$ of $\langle\mathcal{E}_\partial\rangle$ are homotopic, so they are isomorphic as vector bundles and their classes in $K^0(X \times \partial M)$ coincide. This implies $\operatorname{ind}_t(\gamma) = \operatorname{ind}_t(\gamma')$.

(T2). $\mathcal{F}(\gamma \oplus \gamma') = \mathcal{F}(\gamma) \oplus \mathcal{F}(\gamma')$, so $[\mathcal{F}(\gamma \oplus \gamma')] = [\mathcal{F}(\gamma)] + [\mathcal{F}(\gamma')]$ in $K^0(X \times \partial M)$. Applying the homomorphism Ind_t, we obtain the equality $\operatorname{ind}_t(\gamma \oplus \gamma') = \operatorname{ind}_t(\gamma) + \operatorname{ind}_t(\gamma')$ in $K^1(X)$.

(T3). $\mathcal{F}(f^*\gamma) = f^*\mathcal{F}(\gamma)$, so $[\mathcal{F}(f^*\gamma)] = f^*[\mathcal{F}(\gamma)] \in K^0(Y \times \partial M)$. Since the homomorphism $\operatorname{Ind}_t \colon K^0(X \times \partial M) \to K^1(X)$ is natural by X, we have $\operatorname{ind}_t(f^*\gamma) = f^* \operatorname{ind}_t(\gamma)$.

(T4). $\langle\mathcal{F}(1_W \otimes \gamma)\rangle = W \otimes \langle\mathcal{F}(\gamma)\rangle$, so $[\mathcal{F}(1_W \otimes \gamma)] = [W] \cdot [\mathcal{F}(\gamma)] \in K^0(X \times \partial M)$. Both $\alpha_\partial \colon K^0(X \times \partial M) \to K^1(X) \otimes K^1(\partial M)$ and $\operatorname{Id} \otimes \delta \colon K^1(X) \otimes K^1(\partial M) \to K^1(X) \otimes \mathbb{Z}$ are homomorphisms of $K^0(X)$-modules, so their composition $\operatorname{Ind}_t \colon K^0(X \times \partial M) \to K^1(X)$ is also a homomorphism of $K^0(X)$-modules. Combining all this together, we get $\operatorname{ind}_t(1_W \otimes \gamma) = [W] \cdot \operatorname{ind}_t(\gamma)$.

(T5). It is easy to check that, for $X = S^1$ and up to the natural identification $K^1(S^1) \cong \mathbb{Z}$, $\operatorname{Ind}_t[V] = c_1(V)[\partial M \times S^1]$ for every vector bundle V over $\partial M \times S^1$. This implies formula (27.3) and completes the proof of the proposition. □

28 Dirac operators

Recall that k_M denotes the trivial vector bundle over M of rank k with the standard Hermitian structure. Denote by $k_{X,M} \in \mathrm{Vect}_{X,M}$ the trivial bundle over X with the fiber k_M.

Odd Dirac operators. Recall that $\mathrm{Dir}(E)$ denotes the subspace of $\mathrm{Ell}(E)$ consisting of all *odd* Dirac operators, that is, operators having the form

$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix} \text{ with respect to the chiral decomposition } E = E^+(A) \oplus E^-(A),$$

and $\overline{\mathrm{Dir}}(E)$ denotes the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that

$A \in \mathrm{Dir}(E)$. The following two subspaces of $\overline{\mathrm{Dir}}(E)$ will play a special role:

$$\overline{\mathrm{Dir}}^+(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^+(E), \quad \overline{\mathrm{Dir}}^-(E) = \overline{\mathrm{Dir}}(E) \cap \overline{\mathrm{Ell}}^-(E).$$

We denote by $\overline{\mathrm{Dir}}(\mathcal{E})$ the subbundle of $\overline{\mathrm{Ell}}(\mathcal{E})$, whose fiber over $x \in X$ is $\overline{\mathrm{Dir}}(\mathcal{E}_x)$. Similarly, denote by $\overline{\mathrm{Dir}}^+(\mathcal{E})$ and $\overline{\mathrm{Dir}}^-(\mathcal{E})$ the subbundles of $\overline{\mathrm{Dir}}(\mathcal{E})$, whose fibers over $x \in X$ are $\overline{\mathrm{Dir}}^+(\mathcal{E}_x)$ and $\overline{\mathrm{Dir}}^-(\mathcal{E}_x)$, respectively.

Realization of bundles. We will need the following result in our proofs.

Proposition 28.1. *Let $\mathcal{V} \in \mathrm{Vect}_{X,M}$ and let $\mathcal{W}$ be a subbundle of $\mathcal{V}_\partial$. Then there is a section γ of $\overline{\mathrm{Dir}}(\mathcal{V} \oplus \mathcal{V})$ such that $\mathcal{E}^-(\gamma) = \mathcal{V} \oplus 0$ and $\mathcal{F}(\gamma) = \mathcal{W}$. In particular, every vector bundle over $X \times \partial M$ is isomorphic to $\langle \mathcal{F}(\gamma) \rangle$ for some $\gamma \colon X \to \overline{\mathrm{Dir}}(2k_M))$, $k \in \mathbb{N}$.*

Proof. Let us choose smooth global sections e_1, e_2 of TM such that $(e_1(y), e_2(y))$ is a positive oriented frame in T_yM for every $y \in M$. Choose a smooth unitary connection ∇^x on each fiber $\mathcal{V}_x$ in such a way that ∇^x continuously depends on x with respect to the C^1-topology on the space of smooth connections on $\mathcal{V}_x$. (Such a connection can be constructed using a partition of unity subordinated to a finite open covering of X trivializing $\mathcal{V}$.) Then $D_x = -i\nabla^x_{e_1} + \nabla^x_{e_2}$ is the Dirac operator acting on sections of $\mathcal{V}_x$ and depending continuously on x. Let D^t_x be the operator formally adjoint to D_x. Since the operation of taking formally adjoint operator is a continuous transformation of $\mathrm{Ell}(E)$, D^t_x is continuous by x. Thus the operator $A_x = \begin{pmatrix} 0 & D^t_x \\ D_x & 0 \end{pmatrix}$ is an odd self-adjoint Dirac operator acting on sections of $\mathcal{V}_x \oplus \mathcal{V}_x$ and depending continuously on x.

Let T_x be the self-adjoint automorphism of $E^-(A_x) = \mathcal{V}_{\partial,x} \oplus 0$ equal to minus the identity on $\mathcal{W}_x$ and to the identity on the orthogonal complement of $\mathcal{W}_x$ in $\mathcal{V}_x$. Let L_x be the subbundle of $\mathcal{V}_x \oplus \mathcal{V}_x$ corresponding to T_x by formula (15.2). Then $(A_x, L_x) \in \overline{\mathrm{Dir}}(\mathcal{V}_x \oplus \mathcal{V}_x)$ and $F(A_x, L_x) = \mathcal{W}_x$. The section $\gamma \colon x \mapsto (A_x, L_x)$ of $\overline{\mathrm{Dir}}(\mathcal{V} \oplus \mathcal{V})$ satisfies conditions $\mathcal{E}^-(\gamma) = \mathcal{V}$ and $\mathcal{F}(\gamma) = \mathcal{W}$, which proves the first claim of the proposition.

Suppose now that we are given an isomorphism class of a vector bundle over $X \times \partial M$. We can realize it as a subbundle of a trivial vector bundle $k_{X \times \partial M}$ for some $k \in \mathbb{N}$. By Proposition 35.2 from the appendix, this subbundle is homotopic (and thus isomorphic) to $W = \langle \mathcal{W} \rangle$ for some subbundle $\mathcal{W}$ of $k_{X,\partial M}$. Applying conclusion above to $\mathcal{V} = k_{X,M}$ and $\mathcal{W}$, we obtain a section γ of $\overline{\mathrm{Dir}}(\mathcal{V} \oplus \mathcal{V})$ such that $\mathcal{W} = \mathcal{F}(\gamma)$. Since $\mathcal{V} \oplus \mathcal{V} = 2k_{X,M}$ is trivial, γ is just a map from X to $\overline{\mathrm{Dir}}(2k_M)$. This completes the proof of the proposition. □

Image in $K^0(X \times \partial M)$. Denote by $\Gamma^\pm\, \overline{\mathrm{Dir}}(\mathcal{E})$ the subspace of $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$ consisting of sections γ that can be written in the form $\gamma = \gamma' \oplus \gamma''$ with $\gamma' \in \Gamma\overline{\mathrm{Dir}}^+(\mathcal{E}')$ and $\gamma'' \in \Gamma\overline{\mathrm{Dir}}^-(\mathcal{E}'')$ for some orthogonal decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}''$.

Proposition 28.2. *The subgroup G^∂ of $K^0(X \times \partial M)$ is generated by the classes $[\mathcal{F}(\gamma)]$ with γ running over $\Gamma^\pm\, \overline{\mathrm{Dir}}(2k_{X,M})$ and k running over $\mathbb{N}$.*

Proof. The subgroup G^∂ is generated by the images $j^*[V]$ with $V \in \mathrm{Vect}_{X \times M}$. By Proposition 35.2, every such V is isomorphic to $\langle \mathcal{V} \rangle$ for some subbundle $\mathcal{V}$ of $k_{X,M}$ for some (sufficiently large) k. Let $\mathcal{V}'$ be the subbundle of $k_{X,M}$ whose fibers $\mathcal{V}'_x$ are the orthogonal complements of fibers $\mathcal{V}_x$ in k_M. By Proposition 28.1, there are sections $\gamma \in \Gamma\,\overline{\mathrm{Dir}}(\mathcal{V} \oplus \mathcal{V})$ and $\gamma' \in \Gamma\,\overline{\mathrm{Dir}}(\mathcal{V}' \oplus \mathcal{V}')$ such that $\mathcal{E}^-(\gamma) = \mathcal{V}$, $\mathcal{F}(\gamma) = \mathcal{V}_\partial$, $\mathcal{E}^-(\gamma') = \mathcal{V}'$, and $\mathcal{F}(\gamma') = 0$. By Proposition 27.2 $\gamma \in \Gamma\overline{\mathrm{Dir}}^-(\mathcal{V} \oplus \mathcal{V})$ and $\gamma' \in \Gamma\overline{\mathrm{Dir}}^+(\mathcal{V}' \oplus \mathcal{V}')$. Identifying $(\mathcal{V} \oplus \mathcal{V}) \oplus (\mathcal{V}' \oplus \mathcal{V}')$ with $(\mathcal{V} \oplus \mathcal{V}') \oplus (\mathcal{V} \oplus \mathcal{V}') = 2k_{X,M}$, we identify $\gamma \oplus \gamma'$ with an element of $\Gamma^\pm\,\overline{\mathrm{Dir}}(2k_{X,M})$. By construction, $\mathcal{F}(\gamma \oplus \gamma') = \mathcal{V}_\partial \oplus 0$, so $j^*[V] = [\mathcal{V}_\partial] = [\mathcal{F}(\gamma \oplus \gamma')]$. This completes the proof of the proposition. □

Tensor product. Twisting respects Dirac operators and their grading, so its restriction to $\overline{\mathrm{Dir}}(\mathcal{E})$ defines the map $1_W\otimes\colon\ \overline{\mathrm{Dir}}(\mathcal{E}) \to \overline{\mathrm{Dir}}(W \otimes \mathcal{E})$.

Denote by $\Gamma^\boxtimes\,\overline{\mathrm{Dir}}(\mathcal{E})$ the subspace of $\Gamma^\boxtimes\,\overline{\mathrm{Ell}}(\mathcal{E})$ consisting of sections γ having the form $\gamma = \bigoplus_i 1_{W_i} \boxtimes (A_i, L_i)$ for some $(A_i, L_i) \in \overline{\mathrm{Dir}}(E_i)$, $E_i \in \mathrm{Vect}^\infty_M$, and $W_i \in \mathrm{Vect}_X$ with respect to some decomposition of $\mathcal{E}$ into the orthogonal direct sum $\bigoplus_i W_i \boxtimes E_i$.

Proposition 28.3. *The subgroup* $G^\boxtimes$ *of* $K^0(X \times \partial M)$ *is generated by the classes* $[\mathcal{F}(\gamma)]$ *with* γ *running over* $\Gamma^\boxtimes\,\overline{\mathrm{Dir}}(2k_{X,M})$ *and* k *running over* $\mathbb{N}$.

Proof. The subgroup $G^\boxtimes$ is generated by the classes of external tensor products $[W \boxtimes V]$ with $W \in \mathrm{Vect}_X$ and $V \in \mathrm{Vect}^\infty_{\partial M}$. Choose an embedding of W in a trivial vector bundle n_X over X, and let W' be the orthogonal complement of W in n_X. By Proposition 28.1 applied to a one-point base space, we can realize V as $F(A, L)$ for some $(A, L) \in \overline{\mathrm{Dir}}(2k_M)$, $k \in \mathbb{N}$. Choose arbitrary $(A', L') \in \overline{\mathrm{Dir}}^+(2k_M)$. Then $\gamma = 1_W \boxtimes (A, L)$ is a section of $\overline{\mathrm{Dir}}(W \boxtimes 2k_M)$ and $\gamma' = 1_{W'} \boxtimes (A', L')$ is a section of $\overline{\mathrm{Dir}}^+(W' \boxtimes 2k_M)$. Identifying $W \boxtimes 2k_M \oplus W' \boxtimes 2k_M$ with $(W \oplus W') \boxtimes 2k_M = 2nk_{X,M}$, we obtain the section $\gamma \oplus \gamma' \in \Gamma^\boxtimes\,\overline{\mathrm{Dir}}(2nk_{X,M})$ with $\langle \mathcal{F}(\gamma \oplus \gamma') \rangle = (W \boxtimes V) \oplus (W' \boxtimes 0) = W \boxtimes V$. This completes the proof of the proposition. □

Surjectivity of the topological index.

Proposition 28.4. *For every* $\mu \in K^1(X)$ *there are* $k \in \mathbb{N}$ *and* $\gamma\colon X \to \overline{\mathrm{Dir}}(2k_M)$ *such that* $\mu = \mathrm{ind}_t(\gamma)$.

Proof. By Proposition 27.1 the homomorphism $\mathrm{Ind}_t\colon K^0(X \times \partial M) \to K^1(X)$ is surjective, so $\mu = \mathrm{Ind}_t\,\lambda$ for some $\lambda \in K^0(X \times \partial M)$. We can realize λ as $[V] - [n_{X\times\partial M}]$ for some vector bundle V over $X \times \partial M$ and $n \in \mathbb{N}$. By Proposition 28.1 V is isomorphic to $\langle \mathcal{F}(\gamma) \rangle$ for some $\gamma\colon X \to \overline{\mathrm{Dir}}(2k_M)$. The trivial vector bundle $n_{X\times\partial M}$ is the restriction of $n_{X\times M}$ to $X \times \partial M$, so $[n_{X\times\partial M}] \in G^\partial \subset \mathrm{Ker}\,\mathrm{Ind}_t$. Combining all this, we obtain

$$\mathrm{ind}_t(\gamma) = \mathrm{Ind}_t[V] = \mathrm{Ind}_t[V] - \mathrm{Ind}_t[n_{X\times\partial M}] = \mathrm{Ind}_t\,\lambda = \mu.$$

This completes the proof of the proposition. □

29 Universality of the topological index. I

Homotopies that fix operators. In this section we will deal with those deformations of sections of $\overline{\mathrm{Ell}}(\mathcal{E})$ that fix an operator family (A_x) and change only boundary conditions (L_x).

Let us fix an odd Dirac operator $D \in \mathrm{Dir}(2_M)$. Denote by δ^+, respectively δ^-, the constant map from X to $(D, \mathrm{Id}) \in \overline{\mathrm{Dir}}^+(2_M)$, respectively $(D, -\mathrm{Id}) \in \overline{\mathrm{Dir}}^-(2_M)$. We denote by $k\delta^+$, respectively $k\delta^-$, the direct sum of k copies of δ^+, respectively δ^-.

Proposition 29.1. *Let $\gamma\colon x \mapsto (A_x, L_x)$ and $\gamma'\colon x \mapsto (A_x, L'_x)$ be sections of $\overline{\mathrm{Ell}}(\mathcal{E})$ differing only by boundary conditions. Then the following statements hold.*

1. *If $\langle \mathcal{F}(\gamma)\rangle$ and $\langle \mathcal{F}(\gamma')\rangle$ are homotopic subbundles of $\left\langle \mathcal{E}_\partial^-(\gamma)\right\rangle$, then γ and γ' are homotopic sections of $\overline{\mathrm{Ell}}(\mathcal{E})$.*
2. *If $\langle \mathcal{F}(\gamma)\rangle$ and $\langle \mathcal{F}(\gamma')\rangle$ are isomorphic as vector bundles, then the sections $\gamma \oplus k\delta^+$ and $\gamma' \oplus k\delta^+$ of $\overline{\mathrm{Ell}}(\mathcal{E} \oplus 2k_{X,M})$ are homotopic for some $k \in \mathbb{N}$.*
3. *If $[\mathcal{F}(\gamma)] = [\mathcal{F}(\gamma')] \in K^0(X \times \partial M)$, then the sections $\gamma \oplus l\delta^- \oplus k\delta^+$ and $\gamma' \oplus l\delta^- \oplus k\delta^+$ of $\overline{\mathrm{Ell}}(\mathcal{E} \oplus 2l_{X,M} \oplus 2k_{X,M})$ are homotopic for some $l, k \in \mathbb{N}$.*

Proof. Recall that $\mathcal{E}_\partial^-(\gamma)$ depends only on operators, so $\mathcal{E}_\partial^-(\gamma) = \mathcal{E}_\partial^-(\gamma')$; denote it by $\mathcal{E}_\partial^-$. Denote $\mathcal{F} = \mathcal{F}(\gamma)$ and $\mathcal{F}' = \mathcal{F}(\gamma')$.

1. Let $\mathcal{A}\colon x \mapsto A_x$ be the corresponding section of $\mathrm{Ell}(\mathcal{E})$. Denote by $\mathcal{L}(\mathcal{A}) \subset \Gamma\,\overline{\mathrm{Ell}}(\mathcal{E})$ the space of all lifts of $\mathcal{A}$ to sections of $\overline{\mathrm{Ell}}(\mathcal{E})$. Denote by $\mathcal{L}^u(\mathcal{A})$ the subspace of $\mathcal{L}(\mathcal{A})$ consisting of sections (A_x, T_x) such that the self-adjoint automorphisms T_x is unitary for every $x \in X$. The subspace $\mathcal{L}^u(\mathcal{A})$ is a strong deformation retract of $\mathcal{L}(\mathcal{A})$, with the retraction given by the formula $h_s(A_x, T_x) = (A_x, (1 - s + s|T_x|^{-1})T_x)$. Since h_s preserves $\mathcal{F}$, it is sufficient to prove the first claim of the proposition for $\gamma, \gamma' \in \mathcal{L}^u(\mathcal{A})$.

Suppose that $\langle \mathcal{F}\rangle$ and $\langle \mathcal{F}'\rangle$ are homotopic subbundles of $\left\langle \mathcal{E}_\partial^-\right\rangle$. Then $\mathcal{F}$ and $\mathcal{F}'$ are homotopic subbundles of $\mathcal{E}_\partial^-$ by Proposition 35.3 from the appendix. An element $\gamma \in \mathcal{L}^u(\mathcal{A})$ is uniquely defined by the subbundle $\mathcal{F}(\gamma)$ of $\mathcal{E}_\partial^-(\gamma)$. Hence a homotopy between $\mathcal{F}$ and $\mathcal{F}'$ defines the path in $\mathcal{L}^u(\mathcal{A}) \subset \Gamma\,\overline{\mathrm{Ell}}(\mathcal{E})$ connecting γ with γ'.

2. If $\langle \mathcal{F}\rangle$ and $\langle \mathcal{F}'\rangle$ are isomorphic as vector bundles, then they are homotopic as subbundles of $\left\langle \mathcal{E}_\partial^-\right\rangle \oplus k_{X\times\partial M}$ for k large enough. Thus the sections $\gamma \oplus k\delta^+$ and $\gamma' \oplus k\delta^+$ of $\overline{\mathrm{Ell}}(\mathcal{E} \oplus 2k_{X,M})$ satisfy conditions of the first claim of the proposition and therefore are homotopic.

3. The equality $[\mathcal{F}] = [\mathcal{F}']$ implies that the vector bundles $\langle \mathcal{F}\rangle$ and $\langle \mathcal{F}'\rangle$ are stably isomorphic, that is, $\langle \mathcal{F}\rangle \oplus l_{X\times\partial M} = \langle \mathcal{F}(\gamma \oplus l\delta^-)\rangle$ and $\langle \mathcal{F}'\rangle \oplus l_{X\times\partial M} = \langle \mathcal{F}(\gamma' \oplus l\delta^-)\rangle$ are isomorphic for some integer l. It remains to apply the second part of the proposition to the sections $\gamma \oplus l\delta^-$ and $\gamma' \oplus l\delta^-$ of $\overline{\mathrm{Ell}}(\mathcal{E} \oplus 2l_{X,M})$. □

The case of different operators. For a section $\gamma\colon x \mapsto (A_x, T_x)$ of $\overline{\mathrm{Ell}}(\mathcal{E})$ we denote by γ^+ the section of $\overline{\mathrm{Ell}}^+(\mathcal{E})$ given by the rule $x \mapsto (A_x, \mathrm{Id})$.

Proposition 29.2. *Let γ_i be a section of $\overline{\mathrm{Ell}}(\mathcal{E}_i)$, $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Vect}_{X,M}$, $i = 1, 2$. Suppose that $[\mathcal{F}(\gamma_1)] = [\mathcal{F}(\gamma_2)] \in K^0(X \times \partial M)$. Then the sections $\gamma_1 \oplus \gamma_2^+ \oplus l\delta^- \oplus k\delta^+$ and $\gamma_1^+ \oplus \gamma_2 \oplus l\delta^- \oplus k\delta^+$ of $\overline{\mathrm{Ell}}(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus 2l_{X,M} \oplus 2k_{X,M})$ are homotopic for l, k large enough.*

Proof. The sections $\gamma'_1 = \gamma_1 \oplus \gamma_2^+$ and $\gamma'_2 = \gamma_1^+ \oplus \gamma_2$ of $\overline{\mathrm{Ell}}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ differ only by boundary conditions and thus fall within the framework of Proposition 29.1. By

Proposition 27.2 $\mathcal{F}(\gamma_i') = \mathcal{F}(\gamma_i)$. It remains to apply the third part of Proposition 29.1 to γ_1' and γ_2'. □

Commutativity. The direct sum of operators is a non-commutative operation. However, it is commutative up to homotopy, as the following proposition shows.

Proposition 29.3. *Let* $f\colon X \to \overline{\mathrm{Ell}}(2k_M)$, $g\colon X \to \overline{\mathrm{Ell}}(2l_M)$ *be continuous maps. Then* $f \oplus g$ *and* $g \oplus f$ *are homotopic as maps from* X *to* $\overline{\mathrm{Ell}}((2k+2l)_M)$.

Proof. Let J_1 be the unitary automorphism of $\mathbb{C}^{2k+2l}$ given by the formula $u \oplus v \mapsto v \oplus u$ for $u \in \mathbb{C}^{2k}$, $v \in \mathbb{C}^{2l}$. Let us choose a path $J\colon [0,1] \to \mathcal{U}(\mathbb{C}^{2k+2l})$ connecting $J_0 = \mathrm{Id}$ with J_1. Denote by $\tilde{J}_s$ the unitary bundle automorphism of $(2k+2l)_M$ induced by J_s. Then the map $h\colon [0,1] \times X \to \overline{\mathrm{Ell}}((2k+2l)_M)$ defined by the formula $h_s(x) = \tilde{J}_s(f(x) \oplus g(x))$ gives a desired homotopy between $f \oplus g$ and $g \oplus f$. □

Universality of the topological index. Now we are ready to state our first universality result.

Theorem 29.4. *Let* γ_i *be a section of* $\overline{\mathrm{Ell}}(\mathcal{E}_i)$, $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Vect}_{X,M}$, $i = 1, 2$. *Then the following two conditions are equivalent:*

1. $\mathrm{ind}_t(\gamma_1) = \mathrm{ind}_t(\gamma_2)$.
2. *There are* $k, l \in \mathbb{N}$ *and sections* $\beta_i^{\pm} \in \Gamma^{\pm}\, \overline{\mathrm{Dir}}(2k_{X,M})$, $\beta_i^{\boxtimes} \in \Gamma^{\boxtimes}\, \overline{\mathrm{Dir}}(2l_{X,M})$ *such that*

   (29.1) $$\gamma_1 \oplus \gamma_2^{+} \oplus \beta_1^{\pm} \oplus \beta_1^{\boxtimes} \quad \textit{and} \quad \gamma_1^{+} \oplus \gamma_2 \oplus \beta_2^{\pm} \oplus \beta_2^{\boxtimes}$$

   are homotopic sections of $\overline{\mathrm{Ell}}(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus 2k_{X,M} \oplus 2l_{X,M})$.

Proof. $(2 \Rightarrow 1)$ follows immediately from properties (T0–T2) of the topological index.

Let us prove $(1 \Rightarrow 2)$. Suppose that $\mathrm{ind}_t(\gamma_1) = \mathrm{ind}_t(\gamma_2)$. Then $\mathrm{Ind}_t(\lambda_1 - \lambda_2) = 0$ for $\lambda_i = [\mathcal{F}(\gamma_i)] \in K^0(X \times \partial M)$. Proposition 27.1 implies that $\lambda_1 - \lambda_2 = \lambda^{\partial} + \lambda^{\boxtimes}$ for some $\lambda^{\partial} \in G^{\partial}$ and $\lambda^{\boxtimes} \in G^{\boxtimes}$.

By Proposition 28.2 $\lambda^{\partial} = [\mathcal{F}(\beta_2^{\partial})] - [\mathcal{F}(\beta_1^{\partial})]$ for some $\beta_1^{\partial}, \beta_2^{\partial} \in \Gamma^{\pm}\, \overline{\mathrm{Dir}}(2n_{X,M})$ (one can equate the ranks of corresponding trivial bundles by adding several copies of δ^{+} if needed). Similarly, by Proposition 28.3 $\lambda^{\boxtimes} = [\mathcal{F}(\beta_2^{\boxtimes})] - [\mathcal{F}(\beta_1^{\boxtimes})]$ for some $\beta_1^{\boxtimes}, \beta_2^{\boxtimes} \in \Gamma^{\boxtimes}\, \overline{\mathrm{Dir}}(2l_{X,M})$ (one can equate the ranks of corresponding trivial bundles by increasing the ranks of ambient trivial bundles for V and W in construction of $\beta_i^{\boxtimes}$ if needed; see the proof of Proposition 28.3). Combining all this, we obtain

$$\left[\mathcal{F}\left(\gamma_1 \oplus \beta_1^{\boxtimes} \oplus \beta_1^{\partial}\right)\right] = \left[\mathcal{F}\left(\gamma_2 \oplus \beta_2^{\boxtimes} \oplus \beta_2^{\partial}\right)\right].$$

Adding sections of $\overline{\mathrm{Ell}}^{+}(\mathcal{E}_i \oplus 2l_{X,M} \oplus 2n_{X,M})$ to the sections on both sides of this equality, we obtain

$$\left[\mathcal{F}\left(\gamma_1 \oplus \gamma_2^{+} \oplus (\beta_1^{\boxtimes} \oplus \beta_1^{\partial}) \oplus (\beta_2^{\boxtimes} \oplus \beta_2^{\partial})^{+}\right)\right] = \left[\mathcal{F}\left(\gamma_1^{+} \oplus \gamma_2 \oplus (\beta_1^{\boxtimes} \oplus \beta_1^{\partial})^{+} \oplus (\beta_2^{\boxtimes} \oplus \beta_2^{\partial})\right)\right].$$

The third part of Proposition 29.1 implies that

$$\gamma_1 \oplus \gamma_2^+ \oplus (\beta_1^\boxtimes \oplus \beta_1^\partial) \oplus (\beta_2^\boxtimes \oplus \beta_2^\partial)^+ \oplus s\delta^- \oplus t\delta^+$$

and

$$\gamma_1^+ \oplus \gamma_2 \oplus (\beta_1^\boxtimes \oplus \beta_1^\partial)^+ \oplus (\beta_2^\boxtimes \oplus \beta_2^\partial) \oplus s\delta^- \oplus t\delta^+$$

are homotopic for some integers s, t. Using Proposition 29.3 to rearrange terms, taking $k = 2n + l + s + t$, and defining $\beta_i^\pm \in \Gamma^\pm\, \overline{\mathrm{Dir}}(2k_{X,M})$ by the formula

$$\beta_i^\pm = \beta_i^\partial \oplus (\beta_j^\partial \oplus \beta_j^\boxtimes)^+ \oplus s\delta^- \oplus t\delta^+ \quad \text{for } \{i,j\} = \{1,2\},$$

we obtain the second condition of the theorem. □

Universality for families. Our next goal is to describe invariants of families of elliptic operators satisfying the same properties as the topological index. Let $\Phi(\gamma)$ be such an invariant. We start with the first three properties (T0-T2) of the topological index:

($E^\pm$) Φ vanishes on $\Gamma^\pm\, \overline{\mathrm{Ell}}(\mathcal{E})$.

($E^\boxtimes$) Φ vanishes on $\Gamma^\boxtimes\, \overline{\mathrm{Ell}}(\mathcal{E})$.

(E1) $\Phi(\gamma) = \Phi(\gamma')$ if γ and γ' are homotopic sections of $\overline{\mathrm{Ell}}(\mathcal{E})$.

(E2) $\Phi(\gamma \oplus \gamma') = \Phi(\gamma) + \Phi(\gamma')$ for every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ and γ' of $\overline{\mathrm{Ell}}(\mathcal{E}')$.

Let $\mathbb{V}$ be a subclass of $\mathrm{Vect}_{X,M}$ satisfying the following condition:

(29.2) $\quad \mathbb{V}$ is closed under direct sums and contains the trivial bundle $2k_{X,M}$ for every $k \in \mathbb{N}$.

In particular, $\mathbb{V}$ can coincide with the whole $\mathrm{Vect}_{X,M}$.

Theorem 29.5. *Let X be a compact space and Λ be a commutative monoid. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ for every $\mathcal{E} \in \mathbb{V}$. Then the following two conditions are equivalent:*

1. *Φ satisfies properties ($E^\pm$, $E^\boxtimes$, E1, E2) for all $\mathcal{E}, \mathcal{E}' \in \mathbb{V}$;*
2. *Φ has the form $\Phi(\gamma) = \vartheta(\mathrm{ind}_t(\gamma))$ for some (unique) monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$.*

Proof. (2 ⇒ 1) follows immediately from properties (T0–T2) of the topological index.

Let us prove (1 ⇒ 2). We show first that

(29.3) $$\mathrm{ind}_t(\gamma_1) = \mathrm{ind}_t(\gamma_2) \quad \text{implies} \quad \Phi(\gamma_1) = \Phi(\gamma_2)$$

for all $\gamma_i \in \Gamma\overline{\mathrm{Ell}}(\mathcal{E}_i)$, $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{V}$. Indeed, if $\mathrm{ind}_t(\gamma_1) = \mathrm{ind}_t(\gamma_2)$, then by Theorem 29.4 the sections (29.1) are homotopic for some $k, l \in \mathbb{N}$, $\beta_i^{\pm} \in \Gamma^{\pm}\,\overline{\mathrm{Dir}}(2k_{X,M})$, and $\beta_i^{\boxtimes} \in \Gamma^{\boxtimes}\,\overline{\mathrm{Dir}}(2l_{X,M})$. Properties (E1) and (E2) then imply

$$\Phi(\gamma_1) + \Phi(\gamma_2^+) + \Phi(\beta_1^{\pm}) + \Phi(\beta_1^{\boxtimes}) = \Phi(\gamma_2) + \Phi(\gamma_1^+) + \Phi(\beta_2^{\pm}) + \Phi(\beta_2^{\boxtimes}).$$

($E^{\pm}$) implies $\Phi(\gamma_i^+) = \Phi(\beta_i^{\pm}) = 0$, while ($E^{\boxtimes}$) implies $\Phi(\beta_i^{\boxtimes}) = 0$. Thus we obtain $\Phi(\gamma_1) = \Phi(\gamma_2)$, which proves (29.3).

Next we define the homomorphism $\vartheta\colon K^1(X) \to \Lambda$. Let μ be an arbitrary element of $K^1(X)$. By Proposition 28.4 there exist $k \in \mathbb{N}$ and a section β of $\overline{\mathrm{Dir}}(2k_{X,M})$ such that $\mu = \mathrm{ind}_t(\beta)$. In order to satisfy condition (2) of the theorem we have to put $\vartheta(\mu) = \Phi(\beta)$. The correctness of this definition follows from (29.3).

Let now γ be an arbitrary section of $\overline{\mathrm{Ell}}(\mathcal{E})$ and $\mu = \mathrm{ind}_t(\gamma)$. By definition above $\vartheta(\mu) = \Phi(\beta)$ for some β such that $\mu = \mathrm{ind}_t(\beta)$. Then $\mathrm{ind}_t(\gamma) = \mu = \mathrm{ind}_t(\beta)$, so (29.3) implies $\Phi(\gamma) = \Phi(\beta) = \vartheta(\mu) = \vartheta(\mathrm{ind}_t(\gamma))$. This completes the proof of the theorem. □

In the case $\mathbb{V} = \mathrm{Vect}_{X,M}$, the property ($E^{\pm}$) in the statement of the last Theorem 19.3an be replaced by the property ($T^{\pm}$) from the Introduction, namely vanishing of Φ on sections of $\overline{\mathrm{Ell}}^+(\mathcal{E})$ and $\overline{\mathrm{Ell}}^-(\mathcal{E})$. Indeed, a section from $\Gamma^{\pm}\,\overline{\mathrm{Ell}}(\mathcal{E})$ is a sum of sections of $\overline{\mathrm{Ell}}^+(\mathcal{E}')$ and $\overline{\mathrm{Ell}}^-(\mathcal{E}'')$ for some $\mathcal{E}'$ and $\mathcal{E}''$, so ($T^{\pm}$) together with (E2) implies ($E^{\pm}$). Similarly, ($E^{\boxtimes}$) can be replaced by the property ($T^{\boxtimes}$) from the Introduction, namely vanishing of Φ on sections having the form $1_W \boxtimes (A, L)$. Therefore, for $\mathbb{V} = \mathrm{Vect}_{X,M}$ Theorem 29.5 takes the following form, which we give in the Introduction.

Theorem 29.6. *Let X be a compact space and Λ be a commutative monoid. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ for every $\mathcal{E} \in \mathrm{Vect}_{X,M}$. Then the following two conditions are equivalent:*

1. *Φ satisfies properties ($T^{\pm}$, $T^{\boxtimes}$) and (I1, I2).*
2. *Φ has the form $\Phi(\gamma) = \vartheta(\mathrm{ind}_t(\gamma))$ for some (unique) monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$.*

30 Natural transformations of K^1

Let $\mathcal{C}$ be one of the following categories: the category of compact Hausdorff spaces and continuous maps, the category of finite CW-complexes and continuous maps, or the category of smooth closed manifolds and smooth maps. We consider K^1 as a functor from $\mathcal{C}$ to the category of Abelian groups.

The purpose of this section is the proof of the following proposition, which we use in the next section to prove the second collection of results about universality of the family index.

Proposition 30.1. *Let ϑ be a natural self-transformation of the functor $X \mapsto K^1(X)$ respecting the $K^0(\cdot)$-module structure (that is, $\vartheta(\lambda\mu) = \lambda\vartheta(\mu)$ for every object X of $\mathcal{C}$ and every $\lambda \in K^0(X)$, $\mu \in K^1(X)$). Then ϑ is multiplication by some integer c, that is, $\vartheta_X(\mu) = c\mu$ for every object X of $\mathcal{C}$ and every $\mu \in K^1(X)$. In particular, if ϑ_{S^1} is the identity, then ϑ_X is the identity for every X.*

Proof. $K^1(U(1))$ is an infinite cyclic group, so $\vartheta_{U(1)}$ is multiplication by some integer; denote this integer by c.

Let X be an object of $\mathcal{C}$ and $\mu \in K^1(X)$. There is $n \in \mathbb{N}$ and a continuous map $f\colon X \to U(n)$ such that $\mu = f^*\beta$, where β denotes the element of $K^1(U(n))$ corresponding to the canonical representation $U(n) \to \mathrm{Aut}(\mathbb{C}^n)$. Since ϑ is natural, $\vartheta_X\mu = f^*(\vartheta_{U(n)}\beta)$. Therefore, it is sufficient to show that $\vartheta_{U(n)}\beta = c\beta$.

Let $T = U(1)^n$ be the maximal torus in $U(n)$ consisting of diagonal matrices and $V = U(n)/T$ be the flag manifold. Let $\pi\colon V \times T \to U(n)$ be the natural projection given by the formula $\pi(gT, u) = gug^{-1}$.

Denote by $L_1, \dots, L_n$ the canonical linear bundles over V, and let $l_i = [L_i] \in K^0(V)$. Let α_i be the element of $K^1(T)$ corresponding to the projection of $T = U(1)^n$ on the i-th factor. We denote the liftings of L_i, l_i, and α_i to $V \times T$ by the same letters. The lifting of β can be written in these notations as $\pi^*\beta = \sum_{i=1}^n l_i\alpha_i$.

The element α_i is lifted from $U(1)$ and $\vartheta_{U(1)}$ is multiplication by c, hence $\vartheta_{V\times T}(\alpha_i) = c\alpha_i$. Since $\vartheta_{V\times T}$ is a $K^0(V \times T)$-module homomorphism, we have

$$\pi^*(\vartheta_{U(n)}\beta) = \vartheta_{V\times T}(\pi^*\beta) = \sum_{i=1}^n \vartheta_{V\times T}(l_i\alpha_i) = \sum_{i=1}^n l_i \cdot \vartheta_{V\times T}(\alpha_i) = \sum_{i=1}^n l_i \cdot c\alpha_i = \pi^*(c\beta),$$

that is, $\pi^*\left(\vartheta_{U(n)}\beta - c\beta\right) = 0$. To complete the proof of the proposition, it is sufficient to show the injectivity of the homomorphism $\pi^*\colon K^1(U(n)) \to K^1(V \times T)$, which we perform in the following lemma.

Lemma 30.2. *The homomorphism $\pi^*\colon K^*(U(n)) \to K^*(V \times T)$ is injective.*

Proof. The k-th exterior power $U(n) \to \mathrm{Aut}(\Lambda^k\mathbb{C}^n)$ of the canonical representation $U(n) \to \mathrm{Aut}(\mathbb{C}^n)$ defines the element of $K^1(U(n))$; denote this element by β_k. The ring $K^*(U(n))$ is the exterior algebra over $\mathbb{Z}$ generated by $\beta_1, \dots, \beta_n$ [At, Theorem 2.7.17]. Therefore, for every non-zero $\mu \in K^*(U(n))$ there is $\mu' \in K^*(U(n))$ such that $\mu \cdot \mu' = c_\mu b$, where $b = \beta_1 \cdot \ldots \cdot \beta_n$ and c_μ is a non-zero integer. Thus the injectivity of π^* is equivalent to the condition that $c \cdot \pi^*b \neq 0$ in $K^*(V \times T)$ for every integer $c \neq 0$.

By the Künneth formula [At, Theorem 2.7.15], $K^*(T)$ is the exterior algebra over $\mathbb{Z}$ generated by the elements $\alpha_1, \dots, \alpha_n \in K^1(T)$. Applying the Künneth formula one more time, we obtain $K^*(V \times T) = K^*(V) \otimes K^*(T)$. The group $K^*(T)$ is free Abelian and $K^*(V)$ is torsion-free, so $K^*(V) \otimes K^*(T)$ is also torsion-free. Hence we should only prove that $\pi^*b \neq 0$. Let us compute π^*b.

(30.1)

$$\pi^*\beta_k = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} \left(\sum_{i\in I} \alpha_i \cdot \prod_{j\in I} l_j \right) = \sum_{i=1}^n \alpha_i l_i \sum_{\substack{I \subset \{1,\dots,n\}\setminus\{i\} \\ |I|=k-1}} \prod_{j\in I} l_j = \sum_{i=1}^n \alpha_i l_i \left[\Lambda^{k-1}E_i\right],$$

where we denoted $E_i = \bigoplus_{j\neq i} L_j$. Since $[L_i \oplus E_i] = n$, we have $\left[\Lambda^k E_i\right] + l_i \left[\Lambda^{k-1} E_i\right] = \left[\Lambda^k (L_i \oplus E_i)\right] = \binom{n}{k}$, where $\binom{n}{k}$ are the binomial coefficients. Induction by k gives $\left[\Lambda^k E_i\right] = q_k(l_i)$, where the polynomials $q_k \in \mathbb{Z}[x]$ are defined by the formula $q_k(x) = \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} x^j$. Substituting this to (30.1), we get $\pi^*\beta_k = \sum_{i=1}^{n} \alpha_i l_i q_{k-1}(l_i)$. Taking the product of these identities for k running from 1 to n and using the identity $\prod l_i = 1$, we obtain

$$\pi^* b = \prod_{k=1}^{n} \pi^* \beta_k = Q(l_1, \ldots, l_n) \cdot \alpha_1 \cdot \ldots \cdot \alpha_n, \tag{30.2}$$

where $Q \in \mathbb{Z}[x_1, \ldots, x_n]$ is the determinant of the matrix $(q_{k-1}(x_i))_{i,k=1..n}$. Since $(-1)^k q_k(x)$ is a unital polynomial of degree k, the polynomial Q is equal up to sign to the Vandermonde determinant $d_n(x_1, \ldots, x_n) = \det(x_i^{k-1}) = \prod_{i>j}(x_i - x_j)$.

It will be more convenient for us to use $u_k = l_k - 1$ as the generators of $K^0(V)$ instead of l_k. The ring homomorphism $\mathbb{Z}[x_1, \ldots, x_n] \to K^0(V)$ sending x_i to u_i is surjective; its kernel is the ideal J_n generated by the elementary symmetric polynomials $\sigma_k(x_1, \ldots, x_n)$, $k = 1 \ldots, n$ [At, Proposition 2.7.13]. Obviously, $d_n(l_1, \ldots, l_n) = \prod_{i>j}(l_i - l_j) = \prod_{i>j}(u_i - u_j) = d_n(u_1, \ldots, u_n)$.

Let us show that

$$d_n(x_1, \ldots, x_n) \equiv n! \prod_{k=1}^{n-1} x_{k+1}^k \mod J_n. \tag{30.3}$$

Indeed, $d_2(x_1, x_2) = x_2 - x_1 \equiv 2x_2 \mod J_2$. Let $n > 2$ and suppose that

$$d_{n-1}(x_1, \ldots, x_{n-1}) \equiv (n-1)! \prod_{k=1}^{n-2} x_{k+1}^k \mod J_{n-1}. \tag{30.4}$$

Since $\sigma_k(x_1, \ldots, x_{n-1}) + x_n \sigma_{k-1}(x_1, \ldots, x_{n-1}) = \sigma_k(x_1, \ldots, x_n) \equiv 0 \mod J_n$, induction by k implies $\sigma_k(x_1, \ldots, x_{n-1}) \equiv (-1)^k x_n^k \mod J_n$ for all k. Hence

$$\prod_{1\leqslant j\leqslant n-1} (x_n - x_j) = \sum_{k=0}^{n-1} (-1)^k \sigma_k(x_1, \ldots, x_{n-1}) x_n^{n-1-k} \equiv n x_n^{n-1} \mod J_n. \tag{30.5}$$

The inverse image of the ideal J_{n-1} under the projection

$$\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]/(x_n) = \mathbb{Z}[x_1, \ldots, x_{n-1}]$$

is the ideal generated by x_n and J_n. Taking into account induction assumption (30.4), we obtain

$$\prod_{1\leqslant j<i\leqslant n-1} (x_i - x_j) \equiv (n-1)! \prod_{k=2}^{n-1} x_k^{k-1} + x_n f \mod J_n \tag{30.6}$$

for some $f \in \mathbb{Z}[x_1, \dots, x_n]$. Multiplying (30.6) by (30.5), we get

$$\prod_{1 \leqslant j < i \leqslant n} (x_i - x_j) \equiv n! \prod_{k=2}^{n} x_k^{k-1} + nf \cdot x_n^n \mod J_n. \tag{30.7}$$

Since $x_1, \dots, x_n$ are roots of the polynomial $x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n$, their n-th powers x_i^n lie in J_n, so $nf \cdot x_n^n \equiv 0 \mod J_n$, and (30.3) follows from (30.7). Therefore, (30.4) implies (30.3), so (30.3) holds for all $n \geqslant 2$.

The quotient $\mathbb{Z}[x_1, \dots, x_n]/J_n$ is a free Abelian group with the generators $\prod_{k=1}^{n-1} x_{k+1}^{j_k}$, $0 \leqslant j_k \leqslant k$ [Kar, Theorem 3.28]. The right-hand side of (30.3) coincides with one of these generators up to the factor $n!$, so it does not vanish in $\mathbb{Z}[x_1, \dots, x_n]/J_n$. Equivalently, $d_n(u_1, \dots, u_n)$ does not vanish in $K^0(V)$. Taking into account that $\alpha_1 \cdot \ldots \cdot \alpha_n \neq 0$ in $K^*(T)$, we finally obtain

$$\pi^* b = (-1)^{n(n-1)/2} n! \prod_{k=1}^{n-1} u_{k+1}^k \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \neq 0 \text{ in } K^*(V \times T). \tag{30.8}$$

This completes the proof of the lemma and of the proposition. □

31 Universality of the topological index. II

Universality for families: functoriality and twisting. Our next goal is to describe families $\Phi = (\Phi_X)$ of $K^1(X)$-valued invariants satisfying two more properties in addition to ($E^{\pm}$, $E^{\boxtimes}$, E1, E2):

(E3) $\Phi_Y(f_{\mathcal{E}}^* \gamma) = f^* \Phi_X(\gamma) \in K^1(Y)$ for every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ and every continuous map $f\colon Y \to X$.

(E4) $\Phi_X(1_W \otimes \gamma) = [W] \cdot \Phi_X(\gamma)$ for every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ and every $W \in \mathrm{Vect}_X$.

Theorem 31.1. *Suppose that we associate an element $\Phi_X(\gamma) \in K^1(X)$ with every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$ for every compact topological space X and every $\mathcal{E} \in \mathrm{Vect}_{X,M}$. Then the following two conditions are equivalent:*

1. *The family $\Phi = (\Phi_X)$ satisfies properties ($E^{\pm}$, $E^{\boxtimes}$, E1–E4) for all $\mathcal{E}, \mathcal{E}' \in \mathrm{Vect}_{X,M}$;*
2. *There is an integer c such that Φ has the form $\Phi_X = c \cdot \mathrm{ind}_t$.*

Remark 31.2. As well as in Theorem 29.6, the property ($E^{\pm}$) in the statement of this Theorem 19.3an be replaced by ($T^{\pm}$) and ($E^{\boxtimes}$) can be replaced by ($T^{\boxtimes}$).

Proof. (2 ⇒ 1) follows immediately from properties (T0-T4) of the topological index.

Let us prove (1 ⇒ 2). By Theorem 29.5, for every compact space X there is a homomorphism

$$\vartheta_X\colon K^1(X) \to K^1(X) \ \text{ such that } \ \Phi_X(\gamma) = \vartheta_X(\mathrm{ind}_t(\gamma)) \tag{31.1}$$

for every $\mathcal{E} \in \mathrm{Vect}_{X,M}$ and every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$. Moreover, such a homomorphism ϑ_X is unique.

Let $f\colon Y \to X$ be a continuous map and $\mu \in K^1(X)$. By property (T5) of the topological index $\mu = \mathrm{ind}_t(\gamma)$ for some $\gamma \in \Gamma\overline{\mathrm{Ell}}(\mathcal{E})$, $\mathcal{E} \in \mathrm{Vect}_{X,M}$. By (T3) $\mathrm{ind}_t(f^*_{\mathcal{E}}\gamma) = f^* \mathrm{ind}_t(\gamma)$ and by (E3) $\Phi_Y(f^*_{\mathcal{E}}\gamma) = f^*\Phi_X(\gamma)$. Substituting this to (31.1), we obtain

$$\vartheta_Y(f^*\mu) = \vartheta_Y(f^* \mathrm{ind}_t(\gamma)) = \vartheta_Y(\mathrm{ind}_t(f^*_{\mathcal{E}}\gamma)) = \Phi_Y(f^*_{\mathcal{E}}\gamma) = f^*\Phi_X(\gamma) = f^*\vartheta_X \mathrm{ind}_t(\gamma) = f^*\vartheta_X(\mu).$$

Thus the family (ϑ_X) defines a natural transformation ϑ of the functor $X \mapsto K^1(X)$ to itself.

Similarly, (T4) and (E4) imply that ϑ respects the $K^0(\cdot)$-module structure on $K^1(\cdot)$, that is, $\vartheta_X(\lambda\mu) = \lambda\vartheta_X(\mu)$ for every compact space X and every $\lambda \in K^0(X)$, $\mu \in K^1(X)$.

We show in Proposition 30.1 of Part III that the only natural transformations satisfying this property are multiplications by an integer. Hence, there is an integer c such that $\vartheta_X(\mu) = c\mu$ for every X and every $\mu \in K^1(X)$. Substituting this identity to (31.1), we obtain the second condition of the theorem. □

The semigroup of elliptic operators. The disjoint union

$$\overline{\mathrm{Ell}}_M := \coprod_{k\in\mathbb{N}} \overline{\mathrm{Ell}}(2k_M)$$

has the natural structure of a (non-commutative) graded topological semigroup, with the grading by k and the semigroup operation given by the direct sum of operators and boundary conditions. We denote by $\overline{\mathrm{Ell}}_{X,M}$ the trivial bundle over X with the fiber $\overline{\mathrm{Ell}}_M$ and by

$$\Gamma\overline{\mathrm{Ell}}_{X,M} = C(X, \overline{\mathrm{Ell}}_M)$$

the topological semigroup of its sections, with the compact-open topology.

We will use the following two special subsemigroups of $\Gamma\overline{\mathrm{Ell}}_{X,M}$:

$$\Gamma^{\pm}\,\overline{\mathrm{Ell}}_{X,M} = \coprod_{k\in\mathbb{N}} \Gamma^{\pm}\,\overline{\mathrm{Ell}}(2k_{X,M}) \quad \text{and} \quad \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}_{X,M} = \coprod_{k\in\mathbb{N}} \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}(2k_{X,M}).$$

The subsemigroup of $\Gamma\overline{\mathrm{Ell}}_{X,M}$ spanned by $\Gamma^{\pm}\,\overline{\mathrm{Ell}}_{X,M}$ and $\Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}_{X,M}$ will play a special role; we denote it by $\Gamma^{\pm\boxtimes}\,\overline{\mathrm{Ell}}_{X,M}$.

The homotopy classes. The set $\pi_0(\Gamma\overline{\mathrm{Ell}}_{X,M}) = [X, \overline{\mathrm{Ell}}_M]$ of homotopy classes of maps from X to $\overline{\mathrm{Ell}}_M$ has the induced semigroup structure.

Proposition 31.3. *The semigroup* $[X, \overline{\mathrm{Ell}}_M]$ *is commutative for any topological space* X.

Proof. Let $f, g\colon X \to \overline{\mathrm{Ell}}_M$ be continuous maps. For every $k, l \in \mathbb{N}$ the inverse images $f^{-1}(\overline{\mathrm{Ell}}(2k_M))$ and $g^{-1}(\overline{\mathrm{Ell}}(2l_M))$ are open and closed in X, so their intersection $X_{k,l}$ is also open and closed. By Proposition 29.3 the restrictions of $f \oplus g$ and $g \oplus f$ to $X_{k,l}$ are homotopic as maps from $X_{k,l}$ to $\overline{\mathrm{Ell}}((2k+2l)_M)$ (the proof of Proposition 29.3

does not use compactness of X and works as well for arbitrary topological space). Since X is the disjoint union of $X_{k,l}$, this implies that $f \oplus g$ and $g \oplus f$ are homotopic as maps from X to $\overline{\mathrm{Ell}}_M$. Therefore, the classes of $f \oplus g$ and $g \oplus f$ in $[X, \overline{\mathrm{Ell}}_M]$ coincide, so $[X, \overline{\mathrm{Ell}}_M]$ is commutative. This completes the proof of the proposition. □

The topological index as a homomorphism. A continuous map $\gamma\colon X \to \overline{\mathrm{Ell}}_M$ defines the partition of X by subsets X_k, where X_k consists of points X such that $\gamma(x)$ has the grading k. Since the grading is continuous, all X_k are open-and-closed subsets of X. Since X is compact, all but a finite number of X_k are empty, so this partition is finite. The restriction of γ to X_k takes values in $\overline{\mathrm{Ell}}(2k_M)$, so γ can be identified with a section of $\overline{\mathrm{Ell}}(\mathcal{E}_\gamma)$, where $\mathcal{E}_\gamma \in \mathrm{Vect}_{X,M}$ is the bundle whose restriction to X_k is the trivial bundle over X_k with the fiber $2k_M$. Thus the topological index of γ is well defined.

Since the topological index is additive with respect to direct sums, it defines the monoid homomorphism $\mathrm{ind}_t\colon C(X, \overline{\mathrm{Ell}}_M) \to K^1(X)$. Since the topological index is homotopy invariant, this homomorphism factors through the projection $C(X, \overline{\mathrm{Ell}}_M) \to [X; \overline{\mathrm{Ell}}_M]$.

The inclusion $\Gamma^{\pm\boxtimes}\, \overline{\mathrm{Ell}}_{X,M} \hookrightarrow \Gamma\overline{\mathrm{Ell}}_{X,M}$ induces the homomorphism

$$\pi_0(\Gamma^{\pm\boxtimes}\, \overline{\mathrm{Ell}}_{X,M}) \to \pi_0(\Gamma\overline{\mathrm{Ell}}_{X,M}) = [X; \overline{\mathrm{Ell}}_M];$$

we denote its image by $[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes}$.

Since the topological index vanishes on $\Gamma^{\pm}\, \overline{\mathrm{Ell}}_{X,M}$ and $\Gamma^{\boxtimes}\, \overline{\mathrm{Ell}}_{X,M}$, it factors through the quotient $[X; \overline{\mathrm{Ell}}_M]/[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes}$. In other words, there exists a monoid homomorphism

$$\kappa_t\colon [X; \overline{\mathrm{Ell}}_M]/[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes} \to K^1(X)$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} C(X; \overline{\mathrm{Ell}}_M) & \longrightarrow & [X; \overline{\mathrm{Ell}}_M] & \longrightarrow & [X; \overline{\mathrm{Ell}}_M]/[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes} \\ & \searrow^{\mathrm{ind}_t} & & & \downarrow \kappa_t \\ & & & & K^1(X) \end{array} \tag{31.2}$$

Theorem 31.4. *Let* X *be a compact topological space. Then* $[X, \overline{\mathrm{Ell}}_M]/[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes}$ *is an Abelian group isomorphic to* $K^1(X)$*, with an isomorphism given by* κ_t.

Note that, for any given k, the restriction of κ_t to a given rank,

$$[X, \overline{\mathrm{Ell}}(2k_{X,M})]/[X; \overline{\mathrm{Ell}}(2k_{X,M})]^{\pm\boxtimes} \to K^1(X),$$

in general is neither injective nor surjective, so we need to take the direct sum for all the ranks to obtain universality.

Proof. Denote the commutative monoid $[X, \overline{\mathrm{Ell}}_M]/[X; \overline{\mathrm{Ell}}_M]^{\pm\boxtimes}$ by Λ and the composition of horizontal arrows on diagram (31.2) by Φ, so that $\mathrm{ind}_t = \kappa_t \circ \Phi$. By definition,

Φ is additive, homotopy invariant, surjective, and vanishes on both $\Gamma^{\pm}\,\overline{\mathrm{Ell}}_{X,M}$ and $\Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}_{X,M}$.

Suppose first that X is connected. Then $[X, \overline{\mathrm{Ell}}_M] = \coprod_k [X, \overline{\mathrm{Ell}}(2k_M)]$, so Φ and Λ satisfy the first condition of Theorem 29.5 with $\mathbb{V} = \{2k_{X,M}\}$. Thus $\Phi = \vartheta \circ \mathrm{ind}_t$ for some monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$. By Proposition 27.5 the topological index is surjective. Thus κ_t and ϑ are mutually inverse and κ_t is an isomorphism. This completes the proof of the theorem in the case of connected X.

In the general case we need to extend the set $\{2k_{X,M}\}_{k\in\mathbb{N}}$ of trivial bundles. Let $\mathbb{V}$ be the set of all bundles $\mathcal{E}_\gamma$ with $\gamma \in \Gamma\overline{\mathrm{Ell}}_{X,M}$. An element $\mathcal{E}$ of $\mathbb{V}$ is defined by a partition of X by open-and-closed subsets X_k, $k \in \mathbb{N}$, such that all but a finite number of X_k are empty. For such a partition, $\mathcal{E}$ is defined as the disjoint union of trivial bundles $2k_{X_k,M}$. A continuous map from X to $\overline{\mathrm{Ell}}_{X,M}$ is nothing else than a section of a bundle $\overline{\mathrm{Ell}}(\mathcal{E})$ with $\mathcal{E} \in \mathbb{V}$. Obviously, $\mathbb{V}$ is closed under direct sums and contains all trivial bundles $2k_{X,M}$. Hence the triple $(\mathbb{V}, \Phi, \Lambda)$ satisfies the first condition of Theorem 29.5, and therefore $\Phi = \vartheta \circ \mathrm{ind}_t$ for some monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$. Taking into account that both Φ and ind_t are surjective, we see that κ_t and ϑ are mutually inverse and thus κ_t is an isomorphism. This completes the proof of the theorem. □

32 Deformation retraction

Proposition 32.1. *The natural embedding* $\mathrm{Dir}(\mathcal{E}) \hookrightarrow \mathrm{Ell}(\mathcal{E})$ *is a bundle homotopy equivalence for every* $\mathcal{E} \in \mathrm{Vect}_{X,M}$. *Moreover, there exists a fiberwise deformation retraction* h *of* $\mathrm{Ell}(\mathcal{E})$ *onto a subbundle of* $\mathrm{Dir}(\mathcal{E})$ *having the following properties for every* $s \in [0,1]$, $A \in \mathrm{Ell}(\mathcal{E}_x)$, *and* $A_s = h_s(A)$*:*

(1) $E^-(A_s) = E^-(A)$.

(2) The symbol of A_s *depends only on* s *and the symbol* σ_A *of* A.

(3) The map $h'_s\colon \sigma_A \mapsto \sigma_{A_s}$ *defined by (2) is* $U(\mathcal{E}_x)$*-equivariant.*

(4) If $A \in \mathrm{Dir}(\mathcal{E}_x)$, *then* $\sigma_{A_s} = \sigma_A$.

In the case of one-point space X this result was proven in Section 20. We will use Proposition 20.5 to construct such a deformation retraction for an arbitrary compact space X.

Proof. Let (X^i) be a finite open covering of X such that the restrictions of $\mathcal{E}$ to X^i are trivial. Choose trivializations $f^i\colon\ \mathcal{E}|_{X^i} \to E^i \times X^i$. For $x \in X^i$, denote by $f^i_x \in U(\mathcal{E}_x, E^i)$ the isomorphism of the fibers given by f^i. The homeomorphism $\mathrm{Ell}(\mathcal{E}_x) \to \mathrm{Ell}(E^i)$ induced by f^i_x we will also denote by f^i_x.

Choose a partition of unity (ρ_i), $\rho_i \in C(X^i, C^{\infty,1}(M))$, subordinated to the covering (X^i). Let $h^i\colon [0,1] \times \mathrm{Ell}(E^i) \to \mathrm{Ell}(E^i)$ be a deformation retraction of $\mathrm{Ell}(E^i)$ onto a subspace of $\mathrm{Dir}(E^i)$ satisfying conditions of Proposition 20.5.

For $x \in X^i$ and $A \in \mathrm{Ell}(\mathcal{E}_x)$, we define an element A^i_s of $\mathrm{Ell}(\mathcal{E}_x)$ by the formula $f^i_x\left(A^i_s\right) = h^i_s\left(f^i_x(A)\right)$. From Proposition 20.5 we obtain the following:

(a) $A^i_0 = A$ and $A^i_1 \in \mathrm{Dir}(\mathcal{E}_x)$ for every i.

(b) The symbol of A^i_s depends only on s and the symbol σ of A and is independent of i; denote it by σ_s.

(c) The map $\sigma \mapsto \sigma_s$ defined by (b) is $U(\mathcal{E}_x)$-equivariant.

(d) $E^-(\sigma_s) = E^-(\sigma)$.

(e) If $A \in \mathrm{Dir}(\mathcal{E}_x)$, then $\sigma_s = \sigma$ for all $s \in [0,1]$.

(f) If If $A, B \in \mathrm{Ell}(\mathcal{E}_x)$ and the symbols of A^i_1 and B^i_1 coincide, then $A^i_1 = B^i_1$.

We claim that the bundle map $h\colon [0,1] \times \mathrm{Ell}(\mathcal{E}) \to \mathrm{Ell}(\mathcal{E})$ defined by the formula

$$h_s(A) = \sum_i \rho_i(x) A^i_s \quad \text{for } A \in \mathrm{Ell}(\mathcal{E}_x) \tag{32.1}$$

is a desired deformation retraction. The rest of the proof is devoted to the verification of this claim.

First note that (a) implies $h_0 = \mathrm{Id}$. A convex combination of self-adjoint elliptic operators with the symbol σ_s is again a self-adjoint elliptic operator with the symbol σ_s, so (b) implies $\sigma_{A_s} = \sigma_s$ and $A_s \in \mathrm{Ell}(\mathcal{E}_x)$. (c) implies condition (3) of the proposition, (e) implies (4), and (d) implies (1).

The chiral decomposition of an odd Dirac operator A^i_1 is defined by its symbol σ_1 and hence is independent of i, so (a) and (b) imply $\mathrm{Im}\, h_1 \subset \mathrm{Dir}(\mathcal{E})$.

Suppose that $A \in \mathrm{Im}\, h_1$, that is, $A = B_1$ for some $B \in \mathrm{Ell}(\mathcal{E}_x)$. Then $A \in \mathrm{Dir}(\mathcal{E}_x)$, and (e) implies $\sigma_{A_1} = \sigma_A = \sigma_{B_1}$. Hence the symbols of A^i_1 and B^i_1 coincide, and (f) implies $A^i_1 = B^i_1$. Substituting this to (32.1), we obtain $A_1 = B_1$, that is, $h_1(A) = A$. Thus the restriction of h_1 on its image is the identity.

It remains to prove the homotopy equivalence part. Let $A \in \mathrm{Dir}(\mathcal{E}_x)$. Then $A_1 = h_1(A)$ also lies in $\mathrm{Dir}(\mathcal{E}_x)$, but A_s is not necessarily odd for $s \in (0,1)$, so we should change a homotopy a little. Since the symbols of A_1 and A coincide, the formula $h'_s(A) = (1-s)A + sA_1$ defines a continuous bundle map $h'\colon [0,1] \times \mathrm{Dir}(\mathcal{E}) \to \mathrm{Dir}(\mathcal{E})$ such that $h'_0 = \mathrm{Id}$ and $h'_1 = h_1$. It follows that the restriction of h_1 to $\mathrm{Dir}(\mathcal{E})$ and the identity map $\mathrm{Id}_{\mathrm{Dir}(\mathcal{E})}$ are homotopic as bundle maps from $\mathrm{Dir}(\mathcal{E})$ to $\mathrm{Dir}(\mathcal{E})$. On the other hand, the map $h_1\colon \mathrm{Ell}(\mathcal{E}) \to \mathrm{Ell}(\mathcal{E})$ is homotopic to $\mathrm{Id}_{\mathrm{Ell}(\mathcal{E})}$ via the homotopy h_s. It follows that $h_1\colon \mathrm{Ell}(\mathcal{E}) \to \mathrm{Dir}(\mathcal{E})$ is homotopy inverse to the embedding $\mathrm{Dir}(\mathcal{E}) \hookrightarrow \mathrm{Ell}(\mathcal{E})$, that is, this embedding is a bundle homotopy equivalence. This completes the proof of the proposition. □

Proposition 32.2. *For every $\mathcal{E} \in \mathrm{Vect}_{X,M}$ the natural embeddings $\Gamma\,\mathrm{Dir}(\mathcal{E}) \hookrightarrow \Gamma\,\mathrm{Ell}(\mathcal{E})$ and $\Gamma\,\overline{\mathrm{Dir}}(\mathcal{E}) \hookrightarrow \Gamma\,\overline{\mathrm{Ell}}(\mathcal{E})$ are homotopy equivalences. Moreover,*

1. *There exists a deformation retraction of* $\Gamma\, \mathrm{Ell}(\mathcal{E})$ *onto a subspace of* $\Gamma\, \mathrm{Dir}(\mathcal{E})$ *preserving* $\mathcal{E}^-(\gamma)$.

2. *There exists a deformation retraction of* $\Gamma\, \overline{\mathrm{Ell}}(\mathcal{E})$ *onto a subspace of* $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$ *preserving both* $\mathcal{E}^-(\gamma)$ *and* $\mathcal{F}(\gamma)$.

Proof. 1. The fiberwise deformation retraction h from Proposition 32.1 induces the deformation retraction H on the space of sections satisfying conditions of the proposition.

2. Denote by p the natural projection $\Gamma\, \overline{\mathrm{Ell}}(\mathcal{E}) \to \Gamma\, \mathrm{Ell}(\mathcal{E})$, which forgets boundary conditions. We define the deformation retraction $\bar{H}\colon [0,1] \times \Gamma\, \overline{\mathrm{Ell}}(\mathcal{E}) \to \Gamma\, \overline{\mathrm{Ell}}(\mathcal{E})$ by the formula $\bar{H}_s(\gamma)(x) = (H_s(p\gamma)(x), T(x))$ for $\gamma\colon x \mapsto (A(x), T(x))$. Since $\mathcal{E}^-(H_s(p\gamma)) = \mathcal{E}^-(\gamma)$, $\bar{H}_s(\gamma)$ is well defined. By definition of $\bar{H}$, the subbundles $\mathcal{F}(\bar{H}_s(\gamma))$ and $\mathcal{F}(\gamma)$ of $\mathcal{E}^-_\partial(\gamma)$ coincide for every $s \in [0,1]$ and $\gamma \in \Gamma\, \overline{\mathrm{Ell}}(\mathcal{E})$.

3. The fiberwise homotopy h' from Proposition 32.1 induces the homotopy between the restriction of H_1 to $\Gamma\, \mathrm{Dir}(\mathcal{E})$ and the identity map of $\Gamma\, \mathrm{Dir}(\mathcal{E})$, as well as the homotopy between the restriction of $\bar{H}_1$ to $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$ and the identity map of $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$. The same arguments as in the proof of Proposition 32.1 show that $H_1\colon \Gamma\, \mathrm{Ell}(\mathcal{E}) \to \Gamma\, \mathrm{Dir}(\mathcal{E})$ is homotopy inverse to the embedding $\Gamma\, \mathrm{Dir}(\mathcal{E}) \hookrightarrow \Gamma\, \mathrm{Ell}(\mathcal{E})$ and $\bar{H}_1\colon \Gamma\, \overline{\mathrm{Ell}}(\mathcal{E}) \to \Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$ is homotopy inverse to the embedding $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E}) \hookrightarrow \Gamma\, \overline{\mathrm{Ell}}(\mathcal{E})$. This completes the proof of the proposition. □

Retraction of special subspaces. The following proposition is one of the key ingredients in the proof of the index theorem.

Proposition 32.3. *There exists a deformation retraction of* $\Gamma\, \overline{\mathrm{Ell}}^+(\mathcal{E})$ *onto a subspace of* $\Gamma\, \overline{\mathrm{Dir}}^+(\mathcal{E})$ *and a deformation retraction of* $\Gamma\, \overline{\mathrm{Ell}}^-(\mathcal{E})$ *onto a subspace of* $\Gamma\, \overline{\mathrm{Dir}}^-(\mathcal{E})$.

Proof. Let $\bar{H}$ be a deformation retraction of $\Gamma\, \overline{\mathrm{Ell}}(\mathcal{E})$ onto a subspace of $\Gamma\, \overline{\mathrm{Dir}}(\mathcal{E})$ satisfying conditions of Proposition 32.2. For $\gamma \in \Gamma\, \overline{\mathrm{Ell}}^+(\mathcal{E})$ and $\gamma_s = \bar{H}_s(\gamma)$ we have $\mathcal{F}(\gamma_s) = \mathcal{F}(\gamma) = 0$, so by Proposition 27.2 $\gamma_s \in \Gamma\, \overline{\mathrm{Ell}}^+(\mathcal{E})$ for all s. In particular, $\gamma_1 \in \Gamma\, \overline{\mathrm{Ell}}^+(\mathcal{E}) \cap \Gamma\, \overline{\mathrm{Dir}}(\mathcal{E}) = \Gamma\, \overline{\mathrm{Dir}}^+(\mathcal{E})$. For $\gamma \in \Gamma\, \overline{\mathrm{Ell}}^-(\mathcal{E})$ and $\gamma_s = \bar{H}_s(\gamma)$ we have $\mathcal{F}(\gamma_s) = \mathcal{F}(\gamma) = \mathcal{E}^-(\gamma) = \mathcal{E}^-(\gamma_s)$, so by Proposition 27.2 $\gamma_s \in \Gamma\, \overline{\mathrm{Ell}}^-(\mathcal{E})$ for all s. In particular, $\gamma_1 \in \Gamma\, \overline{\mathrm{Ell}}^-(\mathcal{E}) \cap \Gamma\, \overline{\mathrm{Dir}}(\mathcal{E}) = \Gamma\, \overline{\mathrm{Dir}}^-(\mathcal{E})$. This completes the proof of the proposition. □

33 Index theorem

Vanishing of the analytical index. Recall that by Proposition 21.1, A_L has no zero eigenvalues if (A, L) is an element of $\overline{\mathrm{Dir}}^+(E)$ or $\overline{\mathrm{Dir}}^-(E)$. In other words, both $\overline{\mathrm{Dir}}^+(E)$ and $\overline{\mathrm{Dir}}^-(E)$ are subspaces of $\overline{\mathrm{Ell}}^0(E)$. Taking into account Proposition 32.3, we are now able to describe, in purely topological terms, a big class of sections of $\overline{\mathrm{Ell}}(\mathcal{E})$ which are homotopic to families of invertible operators.

Proposition 33.1. *Let* γ *be an element of* $\Gamma^{\pm}\, \overline{\mathrm{Ell}}(\mathcal{E})$ *or* $\Gamma^{\boxtimes}\, \overline{\mathrm{Ell}}(\mathcal{E})$. *Then* γ *is homotopic to a section of* $\overline{\mathrm{Ell}}^0(\mathcal{E})$, *and hence* $\mathrm{ind}_a(\gamma) = 0$.

Proof. 1. If $\gamma \in \Gamma^{\pm}\,\overline{\mathrm{Ell}}(\mathcal{E})$, then $\gamma = \gamma' \oplus \gamma''$ with $\gamma' \in \Gamma\overline{\mathrm{Ell}}^{+}(\mathcal{E}')$ and $\gamma'' \in \Gamma\overline{\mathrm{Ell}}^{-}(\mathcal{E}'')$ for some orthogonal decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}''$. By Proposition 32.3, γ' is homotopic to some $\gamma'_1 \in \Gamma\overline{\mathrm{Dir}}^{+}(\mathcal{E}')$ and γ'' is homotopic to some $\gamma''_1 \in \Gamma\overline{\mathrm{Dir}}^{-}(\mathcal{E}'')$. By Proposition 21.1, γ'_1 and γ''_1 are sections of $\overline{\mathrm{Ell}}^{0}(\mathcal{E}')$ and $\overline{\mathrm{Ell}}^{0}(\mathcal{E}'')$ respectively. It follows that γ is homotopic to $\gamma'_1 \oplus \gamma''_1$, which is a section of $\overline{\mathrm{Ell}}^{0}(\mathcal{E})$.

2. Suppose that $\gamma = 1_W \boxtimes (A, L)$ for some $(A, L) \in \overline{\mathrm{Ell}}(E)$ and $W \in \mathrm{Vect}_X$. Since A_L is Fredholm, $A_L - \lambda$ is invertible for some $\lambda \in \mathbb{R}$, that is, $(A - \lambda, L) \in \overline{\mathrm{Ell}}^{0}(E)$. The path $h\colon [0, 1] \to \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}(W \boxtimes E)$ given by the formula $h_s = 1_W \boxtimes (A - s\lambda, L)$ connects γ with $1_W \boxtimes (A - \lambda, L) \in \overline{\mathrm{Ell}}^{0}(W \boxtimes E)$.

In the general case, for $\gamma = \bigoplus 1_{W_i} \boxtimes (A_i, L_i) \in \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}(\mathcal{E})$, we take such a homotopy as described above for every direct summand $1_{W_i} \boxtimes (A_i, L_i)$ independently. The direct sum of these homotopies gives a required homotopy of γ to a section $h_1(\gamma) \in \Gamma^{\boxtimes}\,\overline{\mathrm{Ell}}^{0}(\mathcal{E})$.

3. It follows from the homotopy invariance of the analytical index and its vanishing on sections of $\overline{\mathrm{Ell}}^{0}(\mathcal{E})$ that $\mathrm{ind}_a(\gamma) = 0$. □

Index theorem. Now we are able to prove our index theorem.

Theorem 33.2. *Let* X *be a compact space and* $\mathcal{E} \in \mathrm{Vect}_{X,M}$. *Then the analytical index is equal to the topological index for every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$*:*

$$\mathrm{ind}_a(\gamma) = \mathrm{ind}_t(\gamma). \tag{33.1}$$

In particular, this equality holds for every continuous map $\gamma\colon X \to \overline{\mathrm{Ell}}(E)$, $E \in \mathrm{Vect}^{\infty}_{M}$.

Proof. By Proposition 8.2 $\Phi = \mathrm{ind}_a$ satisfies conditions (E0–E4). By Proposition 33.1 Φ satisfies conditions ($E^{\pm}$, $E^{\boxtimes}$). By Theorem 31.1 there is an integer c such that $\mathrm{ind}_a(\gamma) = c \cdot \mathrm{ind}_t(\gamma)$ for every section γ of $\overline{\mathrm{Ell}}(\mathcal{E})$, every $\mathcal{E} \in \mathrm{Vect}_{X,M}$, and every compact space X. The factor $c = c_M$ does not depend on X, but can depend on M.

For $X = S^1$ the analytical index of γ coincides with the spectral flow $\mathsf{sf}(\gamma)$ by Proposition 8.2, while the topological index of γ coincides with $c_1(\mathcal{F}(\gamma))[\partial M \times S^1]$ by Proposition 27.5. Hence it is sufficient to compute the quotient

$$c_M = \frac{\mathsf{sf}(\gamma)}{c_1(\mathcal{F}(\gamma))[\partial M \times S^1]}$$

for some loop $\gamma\colon S^1 \to \overline{\mathrm{Dir}}(2k_M)$ such that the denominator of this quotient does not vanish.

It remains to apply Theorem 22.1 to obtain $c_M = 1$ for any surface M. In fact, we do not even need Theorem 22.1 for this, it is enough to use Lemmas 22.4 and 22.5.

Therefore, $\mathrm{ind}_a(\gamma) = \mathrm{ind}_t(\gamma)$, which completes the proof of the theorem. □

34 Universality of the analytical index

Recall that we denoted by $\overline{\mathrm{Ell}}^{0}(E)$ the subspace of $\overline{\mathrm{Ell}}(E)$ consisting of all pairs (A, L) such that the unbounded operator A_L has no zero eigenvalues, and by $\overline{\mathrm{Ell}}^{0}(\mathcal{E})$ the subbundle of $\overline{\mathrm{Ell}}(\mathcal{E})$ whose fiber over $x \in X$ is $\overline{\mathrm{Ell}}^{0}(\mathcal{E}_x)$. Sections of $\overline{\mathrm{Ell}}^{0}(\mathcal{E})$ correspond to families of invertible self-adjoint elliptic boundary problems.

Theorem 34.1. *Let* X *be a compact space, and let* γ_i *be a section of* $\overline{\mathrm{Ell}}(\mathcal{E}_i)$, $\mathcal{E}_i \in \mathrm{Vect}_{X,M}$, $i = 1, 2$. *Then the following two conditions are equivalent:*

1. $\mathrm{ind}_a(\gamma_1) = \mathrm{ind}_a(\gamma_2)$.
2. *There are* $k \in \mathbb{N}$, *sections* β_i^0 *of* $\overline{\mathrm{Ell}}^{0}(2k_{X,M})$, *and sections* γ_i^0 *of* $\overline{\mathrm{Ell}}^{0}(\mathcal{E}_i)$ *such that* $\gamma_1 \oplus \gamma_2^0 \oplus \beta_1^0$ *and* $\gamma_1^0 \oplus \gamma_2 \oplus \beta_2^0$ *are homotopic sections of* $\overline{\mathrm{Ell}}(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus 2k_{X,M})$.

Proof. $(2 \Rightarrow 1)$ follows immediately from properties (I0–I2) of the family index.

Let us prove $(1 \Rightarrow 2)$. By Theorem 33.2 the equality $\mathrm{ind}_a(\gamma_1) = \mathrm{ind}_a(\gamma_2)$ implies $\mathrm{ind}_t(\gamma_1) = \mathrm{ind}_t(\gamma_2)$. By Theorem 29.4 there are $\beta_i^{\pm} \in \Gamma^{\pm}\,\overline{\mathrm{Dir}}(2n_{X,M})$ and $\beta_i^{\boxtimes} \in \Gamma^{\boxtimes}\,\overline{\mathrm{Dir}}(2l_{X,M})$, $i = 1, 2$, such that the direct sums $\gamma_1 \oplus \gamma_2^{+} \oplus \beta_1^{\pm} \oplus \beta_1^{\boxtimes}$ and $\gamma_1^{+} \oplus \gamma_2 \oplus \beta_2^{\pm} \oplus \beta_2^{\boxtimes}$ are homotopic. By Proposition 33.1 γ_i^{+} is homotopic to a section γ_i^0 of $\overline{\mathrm{Ell}}^{0}(\mathcal{E}_i)$ and $\beta_i^{\pm} \oplus \beta_i^{\boxtimes}$ is homotopic to a section β_i^0 of $\overline{\mathrm{Ell}}^{0}(2k_{X,M})$, $k = n + l$. This completes the proof of the theorem. □

Universality for families. In Sections 29 and 31 we considered invariants satisfying properties ($E^{\pm}$, $E^{\boxtimes}$) and (E1-E4). Now we replace the topological properties ($E^{\pm}$, $E^{\boxtimes}$) by the following analytical property:

(E0) Φ vanishes on sections of $\overline{\mathrm{Ell}}^{0}(\mathcal{E})$.

Theorem 34.2. *Let* X *be a compact topological space and* Λ *be a commutative monoid. Let* $\mathbb{V}$ *be a subclass of* $\mathrm{Vect}_{X,M}$ *satisfying condition* (29.2). *Suppose that we associate an element* $\Phi(\gamma) \in \Lambda$ *with every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *for every* $\mathcal{E} \in \mathbb{V}$. *Then the following two conditions are equivalent:*

1. Φ *satisfies properties (E0–E2).*
2. Φ *has the form* $\Phi(\gamma) = \vartheta(\mathrm{ind}_a(\gamma))$ *for some (unique) monoid homomorphism* $\vartheta\colon K^1(X) \to \Lambda$.

Proof. $(2 \Rightarrow 1)$ follows from properties (I0–I2) of the family index. $(1 \Rightarrow 2)$ follows from Theorem 29.4, Proposition 33.1, and Theorem 33.2. □

Theorem 34.3. *Suppose that we associate an element* $\Phi_X(\gamma) \in K^1(X)$ *with every section* γ *of* $\overline{\mathrm{Ell}}(\mathcal{E})$ *for every compact space* X *and every* $\mathcal{E} \in \mathrm{Vect}_{X,M}$. *Then the following two conditions are equivalent:*

1. *The family* $\Phi = (\Phi_X)$ *satisfies properties (E0–E4).*
2. Φ *has the form* $\Phi_X(\gamma) = c \cdot \mathrm{ind}_a(\gamma)$ *for some integer* c.

Proof. (2 ⇒ 1) follows from properties (I0–I4) of the family index. (1 ⇒ 2) follows from Theorem 31.1, Proposition 33.1, and Theorem 33.2. □

Universality for maps. Theorem 34.1 applied to trivial bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ takes the following form.

Theorem 34.4. *Let X be a compact space and $\gamma\colon X \to \overline{\mathrm{Ell}}(2k_M)$, $\gamma'\colon X \to \overline{\mathrm{Ell}}(2k'_M)$ be continuous maps. Then the following two conditions are equivalent:*

1. $\mathrm{ind}_a(\gamma) = \mathrm{ind}_a(\gamma')$.
2. *There are $n \in \mathbb{N}$ and maps $\beta\colon X \to \overline{\mathrm{Ell}}^0(2(n-k)_M)$, $\beta'\colon X \to \overline{\mathrm{Ell}}^0(2(n-k')_M)$ such that the maps $\gamma \oplus \beta$ and $\gamma' \oplus \beta'$ from X to $\overline{\mathrm{Ell}}(2n_M)$ are homotopic.*

Theorem 34.2 applied to the set $\mathbb{V} = \{2k_{X,M}\}$ of trivial bundles takes the following form.

Theorem 34.5. *Let X be a compact space and Λ be a commutative monoid. Suppose that we associate an element $\Phi(\gamma) \in \Lambda$ with every map $\gamma\colon X \to \overline{\mathrm{Ell}}(2k_M)$ for every integer k. Then the following two conditions are equivalent:*

1. *Φ is homotopy invariant, additive with respect to direct sums, and vanishes on maps to $\overline{\mathrm{Ell}}^0(2k_M)$.*
2. *Φ has the form $\Phi(\gamma) = \vartheta(\mathrm{ind}_a(\gamma))$ for some (unique) monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$.*

The analytical index as a homomorphism. Denote by $\overline{\mathrm{Ell}}^0_M$ the disjoint union of subspaces $\overline{\mathrm{Ell}}^0(2k_M) \subset \overline{\mathrm{Ell}}(2k_M)$ for all $k \in \mathbb{N}$; it is a subsemigroup of $\overline{\mathrm{Ell}}_M$. The inclusion $\overline{\mathrm{Ell}}^0_M \subset \overline{\mathrm{Ell}}_M$ induces the homomorphism $[X, \overline{\mathrm{Ell}}^0_M] \to [X, \overline{\mathrm{Ell}}_M]$; we denote by $[X, \overline{\mathrm{Ell}}_M]^0$ its image.

Since the analytical index is additive with respect to direct sums, it defines the monoid homomorphism $\mathrm{ind}_a\colon C(X, \overline{\mathrm{Ell}}_M) \to K^1(X)$. Since the analytical index is homotopy invariant, this homomorphism factors through the homomorphism $C(X, \overline{\mathrm{Ell}}_M) \to [X; \overline{\mathrm{Ell}}_M]$. Since the analytical index vanishes on maps to $\overline{\mathrm{Ell}}^0_M$, it factors through $[X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^0$. In other words, there exists a monoid homomorphism

$$\kappa_a\colon [X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^0 \to K^1(X)$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} C(X, \overline{\mathrm{Ell}}_M) & \longrightarrow & [X, \overline{\mathrm{Ell}}_M] & \longrightarrow & [X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^0 \\ & \searrow^{\mathrm{ind}_a} & & & \downarrow \kappa_a \\ & & & & K^1(X) \end{array} \tag{34.1}$$

Theorem 34.6. *Let X be a compact space. Then $[X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^0$ is an Abelian group isomorphic to $K^1(X)$, and the homomorphism κ_a on diagram (34.1) is an isomorphism.*

Proof. Denote the commutative monoid $[X, \overline{\mathrm{Ell}}_M]/[X, \overline{\mathrm{Ell}}_M]^0$ by Λ and the composition of horizontal arrows on diagram (34.1) by Φ. The homomorphism Φ factor through $[X, \overline{\mathrm{Ell}}_M]$ and vanishes on maps to $\overline{\mathrm{Ell}}^0(2k_M)$. By Theorem 34.5 $\Phi = \vartheta \circ \mathrm{ind}_a$ for some (unique) monoid homomorphism $\vartheta\colon K^1(X) \to \Lambda$. By definition, Φ is surjective. By Theorem 33.2 $\mathrm{ind}_a = \mathrm{ind}_t$; by Proposition 27.5(T5) ind_t is surjective. Thus κ_a and ϑ are mutually inverse monoid homomorphisms, so κ_a is an isomorphism. This completes the proof of the theorem. □

35 Appendix. Smoothing

This appendix discusses forgetting the smooth structure, that is, the correspondence $\mathcal{V} \mapsto \langle\mathcal{V}\rangle$ for $\mathcal{V} \in \mathrm{Vect}_{X,Z}$. The appendix is devoted to the proof of two technical results that are used in the main part of the thesis. Proposition 35.2 states that every vector bundle over $X \times Z$ is isomorphic to $\langle\mathcal{V}\rangle$ for some $\mathcal{V} \in \mathrm{Vect}_{X,Z}$. Proposition 35.3 states that two subbundles $\mathcal{V}_0$ and $\mathcal{V}_1$ of $\mathcal{E}$ are homotopic if the corresponding vector subbundles $\langle\mathcal{V}_0\rangle$ and $\langle\mathcal{V}_1\rangle$ of $\langle\mathcal{E}\rangle$ are homotopic.

Smoothing of maps. Let Z and Z' be compact smooth manifolds and r be a non-negative integer. We denote by $C^{r,\infty}(Z, Z')$ the space $C^\infty(Z, Z')$ of smooth maps from Z to Z' equipped with the topology induced by the natural inclusion $C^\infty(Z, Z') \hookrightarrow C^r(Z, Z')$.

Proposition 35.1. *Let X be a compact space and Z, Z' be compact smooth manifolds. Then for every non-negative integer r the following statements hold:*

1. *The space $C^{r,\infty}(Z, Z')$ is locally contractible.*
2. *The space $C(X \times Z, Z') = C(X, C(Z, Z'))$ is locally contractible and contains $C(X, C^\infty(Z, Z'))$ as a dense subset. In particular, every $f \in C(X, C(Z, Z'))$ is homotopic to some $F \in C(X, C^\infty(Z, Z'))$.*
3. *If continuous maps $f_0, f_1\colon X \to C^{r,\infty}(Z, Z')$ are homotopic as maps from X to $C(Z, Z')$, then they are homotopic as maps from X to $C^{r,\infty}(Z, Z')$. Moreover, $\mathcal{H}^r(f_0, f_1)$ is a dense subset of $\mathcal{H}^0(f_0, f_1)$, where $\mathcal{H}^r(f_0, f_1)$ denotes the subspace of $C([0,1] \times X, C^{r,\infty}(Z, Z'))$ consisting of maps f such that $f|_{\{i\}\times X} = f_i$ for $i = 0, 1$.*

Proof. Let us choose a smooth embedding of Z' in $\mathbb{R}^n$ for some n; let $p\colon N \to Z'$ be its normal bundle. Denote by N_ε the ε-neighborhood of the zero section in N.

Let $\varepsilon > 0$ be small enough, so that the restriction of the geodesic map $q\colon N \to \mathbb{R}^n$ to N_ε is an embedding. This embedding allows to identify N_ε with the ε-neighborhood of Z' in $\mathbb{R}^n$. We denote the restriction of p to N_ε again by p; we will use only this small part of the normal bundle from now on. The map p takes a point $u \in N_\varepsilon$ to the (unique) closest point on Z'.

2a. Let f be an arbitrary element of $C(X \times Z, Z')$. For every $s \in [0,1]$ and every two points $u, v \in Z'$ such that $\|u - v\|_{\mathbb{R}^n} < \varepsilon$, the point $w = su + (1-s)v$ lies in N_ε,

$\|w - p(w)\| = d(w, Z') \leqslant \|w - v\|$, and

$$\|p(w) - u\| = \|p(w) - w + w - u\| \leqslant \|v - w\| + \|w - u\| = \|v - u\| < \varepsilon,$$

so $p(w)$ lies in the ε-neighborhood of u. Thus the formula

$$h_s^i(g) = p \circ (sf + (1-s)g) \tag{35.1}$$

defines the contracting homotopy of the ε-neighborhood

$$U_{f,\varepsilon} = \left\{ g \in C(X \times Z, Z') \colon \|g - f\|_{C(X\times Z, \mathbb{R}^n)} < \varepsilon \right\}$$

of f in $C(X \times Z, Z')$. It follows that $C(X \times Z, Z')$ is locally contractible.

1. If $f \in C(X, C^{r,\infty}(Z, Z'))$, then formula (35.1) defines the contracting homotopy of $C(X, C^{r,\infty}(Z, Z')) \cap U_{f,\varepsilon}$ to f. In the particular case of a one-point space X this implies the first claim of the proposition.

2b. For every $y \in X$ choose $g_y \in C^\infty(Z, \mathbb{R}^n)$ such that $\|g_y - f(y)\|_{C(Z,\mathbb{R}^n)} < \varepsilon$. Then

$$X_y = \left\{ x \in X \colon \|g_y - f(x)\|_{C(Z,\mathbb{R}^n)} < \varepsilon \right\} \tag{35.2}$$

is an open neighborhood of y. Since X is compact, the open covering $(X_y)_{y\in X}$ of X contains a finite sub-covering $(X_y)_{y\in I}$. Choose a partition of unity $(\rho_y)_{y\in I}$ subordinated to this finite covering. We define the map $g' \colon X \to C^\infty(Z, \mathbb{R}^n)$ by the formula $g'(x) = \sum_{y\in I} \rho_y(x) g_y$. Obviously, g' is continuous. By (35.2), $\|g'(x)(z) - f(x)(z)\| < \varepsilon$ for every $x \in X$, $z \in Z$, so the image of g' lies in $C^\infty(Z, N_\varepsilon)$. The composition $g = p \circ g'$ is a continuous map from X to $C^\infty(Z, Z')$. Moreover, g and f are homotopic as continuous maps from $X \times Z$ to Z', with a homotopy given by the formula (35.1). This proves the density of $C(X, C^\infty(Z, Z'))$ in $C(X, C(Z, Z'))$ and completes the proof of the second claim of the proposition.

3. Let $f \colon [0,1] \times X \to C(Z, Z')$ be a homotopy between $f_0, f_1 \in C(X, C^{r,\infty}(Z, Z'))$. By the second claim of the proposition, $C([0,1] \times X, C^\infty(Z, Z'))$ is dense in $C([0,1] \times X, C(Z, Z'))$. Thus there is a continuous map $F \colon [0,1] \times X \to C^{r,\infty}(Z, Z')$ such that $\|F - f\|_{C([0,1]\times X\times Z, \mathbb{R}^n)} < \varepsilon$. The last inequality implies $\|F_i - f_i\|_{C(X\times Z,\mathbb{R}^n)} < \varepsilon$ for $i = 0, 1$, where $F_i = F|_{\{i\}\times X}$. Applying again the second claim of the proposition, we obtain a homotopy $h^{(i)} \colon [0,1] \times X \to C^{r,\infty}(Z, Z')$ between F_i and f_i such that $\left\| h_s^{(i)} - f_i \right\|_{C(X\times Z, \mathbb{R}^n)} < \varepsilon$ for all $s \in [0,1]$. Concatenating $h^{(0)}$, F, and $h^{(1)}$ and suitably reparametrizing the result, we obtain the path in $C(X, C^{r,\infty}(Z, Z'))$ connecting f_0 with f_1 and lying in the ε-neighborhood of f. This proves the third claim of the proposition. □

Smoothing of subbundles. Let us recall some designations from the main part of Part VI. Let X be a topological space and Z be a smooth manifold. We denoted by $\mathrm{Vect}_{X,Z}$ the class of all locally trivial fiber bundles $\mathcal{E}$ over X, whose fiber $\mathcal{E}_x$ is a smooth Hermitian vector bundle over Z for every $x \in X$ and the structure group is the group $U(\mathcal{E}_x)$ of smooth unitary bundle automorphisms of $\mathcal{E}_x$ equipped with the C^1-topology. We say that $\mathcal{W} \subset \mathcal{V}$ is a subbundle of $\mathcal{V} \in \mathrm{Vect}_{X,Z}$ if $\mathcal{W} \in \mathrm{Vect}_{X,Z}$ and $\mathcal{W}_x$ is a smooth subbundle of $\mathcal{V}_x$ for every $x \in X$. For $\mathcal{V} \in \mathrm{Vect}_{X,Z}$ we denoted by $\langle \mathcal{V} \rangle$ the vector bundle over $X \times Z$ whose restriction to $\{x\} \times Z$ is the fiber $\mathcal{V}_x$ with the forgotten smooth structure. Similarly, for a subbundle $\mathcal{W}$ of $\mathcal{V}$ we denote by $\langle \mathcal{W} \rangle$ the corresponding vector subbundle of $\langle \mathcal{V} \rangle$.

Proposition 35.2. *Let X be a compact space, Z be a compact smooth manifold, and V be a subbundle of a trivial vector bundle $k_{X\times Z}$. Then V is homotopic to $\langle \mathcal{V} \rangle$ for some subbundle $\mathcal{V}$ of $k_{X,Z}$. In particular, every vector bundle over $X \times Z$ is isomorphic to $\langle \mathcal{V} \rangle$ for some $\mathcal{V} \in \mathrm{Vect}_{X,Z}$.*

Proof. Let $f\colon X \times Z \to \mathrm{Gr}(\mathbb{C}^k)$ be the continuous map corresponding to the embedding $V \hookrightarrow k_{X\times Z}$. By Proposition 35.1(2), f considered as a map from X to $C(Z, \mathrm{Gr}(\mathbb{C}^k))$ is homotopic to a continuous map $F\colon X \to C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$. Such a map F defines a fiber bundle $\mathcal{V}$ over X, whose fiber $\mathcal{V}_x$ is a smooth subbundle of k_Z given by the smooth map $F(x)\colon Z \to \mathrm{Gr}(\mathbb{C}^k)$. A homotopy between F and f induces the homotopy between the vector subbundles $\langle \mathcal{V} \rangle$ and V of $k_{X\times Z}$.

Let x_0 be an arbitrary point of X and $F_0 = F(x_0)$. By Proposition 35.1(1), there is a contractible neighbourhood U' of F_0 in $C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$. Let h be a corresponding contracting homotopy. Then the restriction of F to $U = F^{-1}(U') \subset X$ is homotopic, as a map from U to $C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$, to the constant map $U \ni x \mapsto F_0$, with the homotopy $H_s(x) = h_s(F(x))$. It follows that the restriction of $\mathcal{V}$ to U is a trivial bundle. Thus $\mathcal{V} \in \mathrm{Vect}_{X,Z}$ and $\mathcal{V}$ is a subbundle of $k_{X,Z}$, which completes the proof of the proposition. □

Proposition 35.3. *Let X and Z be as in Proposition 35.2. Let $\mathcal{E} \in \mathrm{Vect}_{X,Z}$ and $\mathcal{V}_0$, $\mathcal{V}_1$ be subbundles of $\mathcal{E}$. Suppose that $\langle \mathcal{V}_0 \rangle$ and $\langle \mathcal{V}_1 \rangle$ are homotopic as subbundles of $\langle \mathcal{E} \rangle$. Then $\mathcal{V}_0$ and $\mathcal{V}_1$ are homotopic subbundles of $\mathcal{E}$.*

Proof. Consider first the case of a trivial $\mathcal{E} = k_{X,Z}$. Then $\mathcal{V}_i$ can be identified with a continuous map $F_i\colon X \to C^{r,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$, $i = 1, 2$. Since $\langle \mathcal{V}_0 \rangle$ and $\langle \mathcal{V}_1 \rangle$ are homotopic as subbundles of $\langle \mathcal{E} \rangle$, F_0 and F_1 are homotopic as maps from X to $C(Z, \mathrm{Gr}(\mathbb{C}^k))$. By Proposition 35.1(3), they are homotopic as maps from X to $C^{r,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$. It follows that $\mathcal{V}_0$ and $\mathcal{V}_1$ are homotopic subbundles of $\mathcal{E}$.

Let now $\mathcal{E}$ be an arbitrary element of $\mathrm{Vect}_{X,Z}$.

Denote by $\widetilde{\Gamma}\mathcal{E}$ the vector space of continuous maps $X \ni x \mapsto \Gamma^{1,\infty}\mathcal{E}_x$, where $\Gamma^{1,\infty}\mathcal{E}_x$ denotes the space of smooth sections of $\mathcal{E}_x$ with the C^1-topology. It is finitely generated as an $\mathcal{A}$-module, where $\mathcal{A} = C(X, C^{1,\infty}(Z, \mathbb{C}))$. Indeed, let (X_i) be a finite open covering of X such that the restriction $\mathcal{E}_i$ of $\mathcal{E}$ to X_i is a trivial bundle with a fiber E_i. Let (ρ_i) be a partition of unity subordinated to this finite covering, and let (ν_{ij}) be a finite generating set for $\Gamma^\infty E_i$. Then $u_{ij} = \rho_i \nu_{ij}$ form a finite generating set for $\widetilde{\Gamma}\mathcal{E}$.

Let $(u_i)_{i=1}^k$ be a finite generating set for the $\mathcal{A}$-module $\widetilde{\Gamma}\mathcal{E}$. For every $x \in X$, the set $(u_i(x))$ of smooth sections of $\mathcal{E}_x$ generates $\Gamma^\infty\mathcal{E}_x$ as a $C^\infty(Z, \mathbb{C})$-module and thus defines the smooth surjective bundle morphism $\pi_x\colon k_Z \to \mathcal{E}_x$ continuously depending on x. Then the kernel $\mathcal{K}_x$ of π_x continuously depends on x and is locally trivial. Thus the family $(\mathcal{K}_x)$ of smooth vector subbundles of k_Z defines the subbundle $\mathcal{K}$ of $k_{X,Z}$. Denote by K the continuous map from X to $C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$ corresponding to $\mathcal{K}$. Obviously, subbundles of $\mathcal{E}$ are in one-to-one correspondence with subbundles of $k_{X,Z}$ containing $\mathcal{K}$.

Let $\mathcal{V}_0$, $\mathcal{V}_1$ be subbundles of $\mathcal{E}$. Denote by $\mathcal{W}_0$, $\mathcal{W}_1$ the corresponding subbundles of $k_{X,Z}$ and by F_0, F_1 the corresponding maps from X to $C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$. If $\langle \mathcal{V}_0 \rangle$ and $\langle \mathcal{V}_1 \rangle$ are homotopic as subbundles of $\langle \mathcal{E} \rangle$, then there is a homotopy $h\colon [0,1] \times X \to C(Z, \mathrm{Gr}(\mathbb{C}^k))$ between F_0 and F_1 such that $h_s(x)(z) \supset K(x)(z)$ for every $s \in [0,1]$, $x \in X$, and $z \in Z$.

Equip $\mathrm{Gr}(\mathbb{C}^k)$ with a smooth Riemannian metric. For $L \in \mathrm{Gr}(\mathbb{C}^k)$ denote by $\mathrm{Gr}_L(\mathbb{C}^k)$ the submanifold of $\mathrm{Gr}(\mathbb{C}^k)$ consisting of subspaces of $\mathbb{C}^k$ containing L. Denote by $p_L\colon N_L \to \mathrm{Gr}_L(\mathbb{C}^k)$ the normal bundle of $\mathrm{Gr}_L(\mathbb{C}^k)$ in $\mathrm{Gr}(\mathbb{C}^k)$, and by $N_{L,\varepsilon}$ the ε-neighborhood of the zero section in N_L. Let $\varepsilon > 0$ be small enough, so that for every $L \in \mathrm{Gr}(\mathbb{C}^k)$ the geodesic map $q_L\colon N_{L,\varepsilon} \to \mathrm{Gr}(\mathbb{C}^k)$ is an embedding. Similarly to the proof of Proposition 35.1, we identify $N_{L,\varepsilon}$ with the ε-neighborhood of $\mathrm{Gr}_L(\mathbb{C}^k)$ in $\mathrm{Gr}(\mathbb{C}^k)$. The map p_L smoothly depends on L with respect to this identification.

By Proposition 35.1(3), there is a homotopy $H\colon [0,1] \times X \to C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$ between F_0 and F_1 such that the distance between $H_s(x)(z)$ and $h_s(x)(z)$ is less then ε for all s, x, and z. Then the continuous map $F\colon [0,1] \times X \to C^{1,\infty}(Z, \mathrm{Gr}(\mathbb{C}^k))$ defined by the formula $F_s(x)(z) = p_{K(x)(z)}(H_s(x)(z))$ is a homotopy between F_0 and F_1 such that $F_s(x)(z) \supset K(x)(z)$ for every s, x, and z. Thus F defines the homotopy $(\mathcal{W}_s)$ between $\mathcal{W}_0$ and $\mathcal{W}_1$ such that $\mathcal{K}$ is a subbundle of $\mathcal{W}_s$ for every $s \in [0,1]$. Factoring by $\mathcal{K}$, we obtain the homotopy $(\mathcal{V}_s)$ between $\mathcal{V}_0$ and $\mathcal{V}_1$ as subbundles of $\mathcal{E}$, which completes the proof of the proposition. □

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Appendix A
Dirac operators on planar domains

In this appendix we apply Theorem 23.1 to Dirac operators on a planar domain. Our aim is to provide the reader with the simplest and most basic examples. In addition, these examples may be useful for condensed matter physics; one of the possible applications is to the Aharonov-Bohm effect for a single-layer graphene sheet with holes.

In these examples, we consider a one-parameter family (D_t) of Dirac operators on a planar domain. We suppose that all the operators D_t have the same symbol and that D_0 and D_1 are conjugated by a scalar gauge transformation. We consider all the operators D_t with the same boundary condition.

In the last section of the appendix, we compute the spectral flow for 4-dimensional Dirac operators in terms of the "general boundary conditions for the Dirac equation" formulated by Akhmerov and Beenakker in [AB].

All these examples are taken from the author's earlier paper [P1]. However, in [P1] the spectral flow was computed explicitly only for the case of an annulus. The spectral flow formula for a disk with more than one hole was obtained there only up to a multiplicative integer constant. In Part V of the thesis we compute this constant explicitly for the general case, which allows us to give an exact value of the spectral flow for all the examples considered in the appendix.

36 Dirac operators: the simplest case

Let M be a compact planar domain bounded by m smooth curves (topologically it is a disk with $m-1$ holes).

Let $\mathbb{D}$ be the Dirac operator on M,

$$\mathbb{D} = -i \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}, \tag{36.1}$$

where ∂_i denotes the partial derivative $\partial/\partial x^i$ and $x = (x^1, x^2)$ are coordinates on $M \subset \mathbb{R}^2$. $\mathbb{D}$ acts on spinor-valued functions, which we identify with column vectors of two complex-valued functions:

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix},\ u^\pm \colon M \to \mathbb{C}.$$

A Dirac operator with non-zero vector potential has the form

$$D = \mathbb{D} + Q,\ \text{where } Q = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}$$

and q is a smooth function from M to $\mathbb{C}$.

Let $g \colon M \to U(1) = \{z \in \mathbb{C} \colon |z| = 1\}$ be a scalar gauge transformation. The conjugation by g takes the Dirac operator $D_0 = \mathbb{D} + Q_0$ to the Dirac operator $D_1 = gD_0g^{-1} =$

$\mathbb{D} + Q_1$. Let

$$D_t = \mathbb{D} + Q_t, \quad Q_t = \begin{pmatrix} 0 & \overline{q}_t \\ q_t & 0 \end{pmatrix}, \tag{36.2}$$

be a one-parameter family of Dirac operators connecting D_0 with D_1. In other words, q_t is a smooth function from M to $\mathbb{C}$ continuously depending on $t \in [0,1]$ and satisfying the conjugation condition

$$q_1 - q_0 = ig^{-1}(\partial_1 g + i\partial_2 g).$$

A self-adjoint elliptic local boundary condition for D_t has the form

$$i(n_1 + in_2)u^+ = Tu^- \text{ on } \partial M, \tag{36.3}$$

where $T\colon \partial M \to \mathbb{R} \setminus \{0\}$ is a smooth function defining the boundary condition and $n = (n_1, n_2)$ is the outward normal to the boundary ∂M at point x.

Remark 36.1. A local boundary condition for D_t is elliptic if and only if it can be written in the form (36.3) with $T\colon \partial M \to \mathbb{C} \setminus \{0\}$. Such a boundary condition is self-adjoint if and only if the function T is real-valued.

Remark 36.2. $D = \mathbb{D} + Q$ is the Dirac operator on the trivial 2-dimensional complex vector bundle over M with the compatible unitary connection defined by the function $q\colon M \to \mathbb{C}$. So the change of q_t with t is equivalent to a change of the connection.

The boundary condition (36.3) is gauge invariant with respect to conjugation by g, while D_0 and D_1 are conjugated by g. So the operators D_0, D_1 with the same boundary condition (36.3) are isospectral, and the spectral flow of the family D_t determines a shift of the spectrum of D_t when t runs from 0 to 1.

Proposition 36.3. *The spectral flow of the family* (D_t), $t \in [0,1]$, *with boundary condition* (36.3) *is given by the formula*

$$\mathsf{sf}(D_t, T) = \sum_{j=1}^{m} \varepsilon_j g_j,$$

where g_j *is the degree of the restriction of* g *to* ∂M_j *and*

$$\varepsilon_j = \begin{cases} 1, & \textit{if } T < 0 \textit{ on } \partial M_j \\ 0, & \textit{if } T > 0 \textit{ on } \partial M_j \end{cases}.$$

Here ∂M_j *are the connected components of the boundary of* M *equipped with an orientation in such a way that the pair (outward normal to* ∂M_j*, positive tangent vector to* ∂M_j*) has positive orientation on the plane* (x^1, x^2).

Note that since T does not take zero values, it has a definite sign at each boundary component ∂M_j, so the constants ε_j are well defined. The restriction of g to the j-th connected component of ∂M gives us a map from the circle ∂M_j to the circle $U(1)$; g_j is the degree of this map.

Proof. This is an immediate corollary of Theorem 23.1. □

Remark 36.4. Boundary condition (36.3) coincides with the boundary condition of Berry and Mondragon for the "neutrino billiard" [Be] up to replacement of B by $1/T$. In physical terms, a one-parameter family of Dirac operators (36.2) describes the situation of a magnetic field varying continuously so that the following two conditions are fulfilled:

(1) the magnetic field at $t = 1$ coincides with the magnetic field at $t = 0$ all over the interior of M,

(2) the fluxes through the j-th hole at $t = 1$ and at $t = 0$ differ by an integer number g_j in the units of the flux quantum.

Suppose that $j = m$ corresponds to the outer boundary component and that $j = 1, \dots, m-1$ enumerate the holes. Since $g_m = -\sum_{j=1}^{m-1} g_j$, we can reformulate Proposition 36.3 as follows. The spectral flow of the family of operators (36.2) with boundary condition (36.3) is given by the formula

$$\operatorname{sf}(D_t, T) = \sum_{j=1}^{m-1} (\varepsilon_j - \varepsilon_m) g_j. \tag{36.4}$$

Thus the variation of the magnetic field through the j-th hole contributes to the value of the spectral flow with the coefficient $(\varepsilon_j - \varepsilon_m)$.

In particular, in the case of one hole equality (36.4) takes the form

$$\operatorname{sf}(D_t, T) = (\varepsilon_1 - \varepsilon_2) \deg(g).$$

If the signs of T are the same on all boundary components, then the spectral flow is zero, no matter how the magnetic field is varied (if only conditions (1-2) above are fulfilled). On the contrary, if T takes positive values on some boundary component and negative values on another, then one can vary the magnetic field so that the spectral flow does not vanish.

37 2k-dimensional Dirac operators

Let M be as in the previous section. The standard 2k-dimensional Dirac operator has the form

$$\mathbb{D} = -i\left(\sigma_1 \partial_1 + \sigma_2 \partial_2\right), \quad \text{where } \sigma_1 = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -iI_k \\ iI_k & 0 \end{pmatrix}, \tag{37.1}$$

and I_k is the $k \times k$ unit matrix.

We will consider operators of the form $D = \mathbb{D} + Q(x)$ acting on spinor functions

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \ u^\pm \colon M \to \mathbb{C}^k, \tag{37.2}$$

where Q is a smooth map from M to the space $\mathcal{B}^{sa}(\mathbb{C}^{2k})$ of complex self-adjoint $2k \times 2k$ matrices.

A self-adjoint elliptic local boundary condition for the operator $\mathbb{D} + Q$ has the form

$$i(n_1 + in_2)u^+ = Tu^- \quad \text{on } \partial M, \tag{37.3}$$

where T is a smooth map from ∂M to the space of complex self-adjoint invertible $k \times k$ matrices and $n(x) = (n_1, n_2)$ is the outward conormal to ∂M at point x.

The equivalent representation of boundary condition (37.3) is

$$\left(i\left(n_1\sigma_1 + n_2\sigma_2\right) + \begin{pmatrix} T^{-1} & 0 \\ 0 & -T \end{pmatrix}\right) u = 0 \quad \text{on } \partial M. \tag{37.4}$$

Remark 37.1. A local boundary condition for D_t is elliptic if and only if it can be written in the form (36.3) with $T(x)$ invertible for every x; such a boundary condition is self-adjoint if and only if $T(x)$ is self-adjoint for every x.

Proposition 37.2. *Let $Q_t(x)$ be a continuous one-parameter family of self-adjoint $2k \times 2k$ matrices smoothly depending on $x \in M$ such that $\mathbb{D} + Q_1 = g\,(\mathbb{D} + Q_0)\,g^{-1}$ for some smooth gauge transformation $g\colon M \to U(1)$. Let T be a smooth map from ∂M to the space of complex self-adjoint invertible $k \times k$ matrices. Then the spectral flow of the family $(\mathbb{D} + Q_t)$ with boundary condition (37.3) is given by the formula*

$$\mathsf{sf}\,(\mathbb{D} + Q_t, T) = \sum_{j=1}^{m} \varepsilon_j g_j,$$

where g_j is the degree of the restriction of g to the j-th connected component ∂M_j of the boundary and ε_j is the number of negative eigenvalues of T (counting multiplicities) on ∂M_j (this number is correctly defined due to the nondegeneracy of T).

Proof. This is an immediate corollary of Theorem 23.1. □

38 The spectral flow for $k = 2$ in terms of condensed matter physics

In this section we compare our boundary condition (37.3) with the "general boundary conditions" for 4-dimensional Dirac operators formulated by Akhmerov and Beenakker in [AB] and give some computations for the spectral flow in terms of [AB].

In this section we will temporarily use notations of [AB] in their original form and will formulate our results in the same terms.

The long-wavelength and low-energy electronic excitations in graphene (a one-atom-thick planar sheet of single carbon atoms that are densely packed in a honeycomb crystal lattice) considered in [AB] are described by the Dirac equation $H\Psi = E\Psi$ with the Hamiltonian

$$H = v\tau_0 \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{p}) \tag{38.1}$$

acting on a four-component spinor wave function $\Psi = (\Psi_A, \Psi_B)$ (in our notations, Ψ is a two-dimensional spinor function, $\Psi_A = u^+$, $\Psi_B = u^-$, $k = 2$). Here v is the Fermi velocity, $\boldsymbol{p} = -i\hbar\nabla$ is the momentum operator, $\boldsymbol{\sigma} \cdot \boldsymbol{p} = -i\hbar(\sigma_1\nabla_1 + \sigma_2\nabla_2)$, matrices τ_i, σ_i are Pauli matrices in valley space and sublattice space, respectively:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \tau_i = \sigma_i.$$

The general energy-independent boundary condition posed in [AB] has the form

$$\Psi = \mathcal{M}\Psi \text{ on the boundary,} \tag{38.2}$$

where $\mathcal{M}$ is a self-adjoint unitary 4×4 matrix depending on the point $x \in \partial M$ and anticommuting with the current operator $v\tau_0 \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{n}_B)$. Here $\boldsymbol{n}_B$ is the outward normal to the boundary of M at x, so $\boldsymbol{n}_B = (n_1, n_2)$ in our previous notations, and $\boldsymbol{\sigma} \cdot \boldsymbol{n}_B = n_1\sigma_1 + n_2\sigma_2$.

Let us compare (38.2) with our boundary condition (37.3).

At first note that the condition "$\mathcal{M}$ is a self-adjoint unitary matrix anticommuting with the current operator" means nothing but the condition of self-adjointness of the boundary problem (38.2). The authors of [AB] do not require local ellipticity of the boundary condition; however, in the absence of local ellipticity the spectrum of the operator is no longer expected to be discrete. The boundary condition (38.2) is both locally elliptic and self-adjoint if and only if the matrix function $\mathcal{M}$ can be represented by the formula

$$\mathcal{M} = I_{2k} - 2\begin{pmatrix} I_k + T^2 & 0 \\ 0 & I_k + T^2 \end{pmatrix}^{-1} \begin{pmatrix} I_k & i\bar{n}T \\ -inT & T^2 \end{pmatrix}$$

for some complex self-adjoint invertible $k \times k$ matrix function T. In this case, boundary condition (38.2) is equivalent to our boundary condition (37.4).

The set of all possible self-adjoint unitary 4×4 matrices anticommuting with the current operator is parametrized in [AB] by the following 4-parameter family:

$$\mathcal{M} = \sin\Lambda \ \nu_0 \otimes (\boldsymbol{n}_1 \cdot \boldsymbol{\sigma}) + \cos\Lambda \ (\boldsymbol{\nu} \cdot \boldsymbol{\tau}) \otimes (\boldsymbol{n}_2 \cdot \boldsymbol{\sigma}), \tag{38.3}$$

where $\Lambda(x)$ is the "mixing angle", $\boldsymbol{\nu}(x)$, $\boldsymbol{n}_1(x)$, $\boldsymbol{n}_2(x)$ are unit vectors in $\mathbb{R}^3 = \{(x^1, x^2, x^3)\}$ such that $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are mutually orthogonal and also orthogonal to the boundary normal $\boldsymbol{n}_B(x)$, $(\boldsymbol{\nu} \cdot \boldsymbol{\tau}) = \sum_{i=1}^{3} \nu^i\tau_i$, and $(\boldsymbol{n}_j \cdot \boldsymbol{\sigma})$ are defined analogously.

Let us describe the ellipticity condition for the boundary problem (38.2) in terms of $(\Lambda, \boldsymbol{\nu}, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and compute ε_j as functions of these parameters.

From now on we will suppose that the frame $(\boldsymbol{n}_B, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is *positively oriented* in $\mathbb{R}^3$, that is its orientation coincides with the orientation of the frame (e_1, e_2, e_3) of basis coordinate vectors. This is possible because parameters $(\Lambda, \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\nu})$ and $(-\Lambda, -\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\nu})$ define the same matrix $\mathcal{M}$, so in the case of a negatively oriented frame $(\boldsymbol{n}_B, \boldsymbol{n}_1, \boldsymbol{n}_2)$ we can change the signs of $\boldsymbol{n}_1$ and Λ simultaneously to obtain the positive orientation of the frame.

Let φ be a function from the boundary to the circle $\mathbb{R} \bmod 2\pi$ such that $\boldsymbol{n}_2 = \sin\varphi \cdot \eta + \cos\varphi \cdot e_3$, where e_3 is the unit vector in $\mathbb{R}^3$ in the direction of x^3, and $\eta(x)$ is the unit tangent vector to the boundary at $x \in \partial M$ such that the pair $(\boldsymbol{n}_B(x), \eta(x))$ has the positive orientation on the plane (x^1, x^2). Then $\boldsymbol{n}_1 = \cos\varphi \cdot \eta - \sin\varphi \cdot e_3$, and $\mathcal{M}$ is determined by the triple $(\Lambda, \varphi, \boldsymbol{\nu})$.

The following two propositions are Propositions 1-2 from [P1].

Proposition 38.1. *The boundary condition* (38.2) *is locally elliptic for the Dirac operator* (38.1) *if and only if* $\Lambda + \varphi \neq 0 \pmod{\pi}$ *and* $\Lambda - \varphi \neq 0 \pmod{\pi}$ *at every point of the boundary* ∂M.

In other words, the boundary condition is *not* locally elliptic if and only if $\boldsymbol{n}_2 = \pm\sin\Lambda\,\eta \pm \cos\Lambda\, e_3$ for some $x \in \partial M$ and for some combination of signs $\pm$.

Proposition 38.2. *If the boundary condition* (38.2) *is locally elliptic for the Dirac operator* (38.1)*, then it is equivalent to boundary condition* (37.3) *with* $T = \mu_+ P_+ + \mu_- P_-$*, where*

$$P_\pm = \frac{\mu_0 \pm (\boldsymbol{\nu} \cdot \boldsymbol{\tau})}{2}, \; \mu_+ = \cot\frac{\Lambda + \varphi}{2}, \; \mu_- = \tan\frac{\Lambda - \varphi}{2}.$$

Here $\mu_\pm$ *are the eigenvalues of* T *and* $P_\pm$ *are the orthogonal projections on the invariant subspaces of* T *corresponding to the eigenvalues* $\mu_\pm$*.*

Proposition 38.3. *Let* $Q_t(x)$ *be a continuous one-parameter family of self-adjoint* 4×4 *matrices smoothly depending on* $x \in M$ *such that* $H + Q_1 = g\,(H + Q_0)\,g^{-1}$ *for some smooth gauge transformation* $g \colon M \to U(1)$*. Suppose that the boundary condition* (38.2) *is locally elliptic for the Dirac operator* (38.1)*. Then the spectral flow of the family* $(H + Q_t)$ *with this boundary condition is described by the formula*

$$\mathsf{sf}\,(H + Q_t, \mathcal{M})_{t \in [0,1]} = \sum_{j=1}^{m} \varepsilon_j g_j,$$

where g_j *is the degree of the restriction of* g *to* ∂M_j *and* ε_j *depends only on the values of* Λ*,* φ *at the* j*-th boundary component* ∂M_j*:*

$$\varepsilon_j = \begin{cases} 0, & \text{if both } \Lambda + \varphi,\ \Lambda - \varphi \text{ belong to the interval } (0, \pi) \\ 2, & \text{if both } \Lambda + \varphi,\ \Lambda - \varphi \text{ belong to the interval } (\pi, 2\pi) \\ 1, & \text{if one of } \Lambda + \varphi,\ \Lambda - \varphi \text{ belongs to the interval } (0, \pi) \text{ and another to the interval } (\pi, 2\pi) \end{cases}$$

Proof. This proposition follows immediately from Propositions 37.2 and 38.2. □

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הזרימה הספקטרלית ואינדקס המשפחה
עבור בעיות ערכי שפה אליפטיות הצמודות לעצמן
מעל משטחים קומפקטיים

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר דוקטור לפילוסופיה

מרינה פרוחורוב

הוגש לסנט הטכניון – מכון טכנולוגי לישראל

תמוז תש"ף   חיפה   יוני 2020

המחקר נעשה בהנחיית פרופסור אמריטוס שמעון רייך בפקולטה למתמטיקה.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי.

תקציר

החיבור עוסק באופרטורים דיפרנציאליים אליפטיים הצמודים לעצמם מסדר ראשון מעל משטח חלק קומפקטי עם אוריינטציה ועם שפה. אנו חוקרים את האופרטורים האלה עם תנאיי שפה מקומיים אליפטיים הצמודים לעצמם.

החלק הראשון של התוצאות נוגע למסילות במרחב של בעיות ערכי שפה המחברות בין שתי בעיות ערכי שפה שצמודות על ידי אוטומורפיזם אוניטרי. אנו מחשבים את הזרימה הספקטרלית עבור המסילות האלה מבחינת הנתונים הטופולוגיים בשפה.

החלק השני של התוצאות נוגע למשפחות של בעיות ערכי שפה התלויות בפרמטר הרץ בתוך מרחב טופולוגי קומפקטי כלשהו X. אנו מוכיחים משפט אינדקס משפחה עבור משפחות כאלה, כלומר מחשבים את האינדקס האנליטי של משפחה המוערך ב- $K^1(X)$ באמצעות הנתונים הטופולוגיים בשפה.

המרחב של בעיות ערכי שפה.

יהי M משטח חלק קומפקטי עם אוריינטציה ועם שפה ∂M, ויהי E אגד ווקטורי הרמיטי מעל M. נסמן על ידי $\overline{Ell}(E)$ את המרחב של כל הזוגות (A, L) כך ש-A הוא אופרטור דיפרנציאלי אליפטי הצמוד לעצמו מסדר ראשון הפועל על החתכים של E ו-L הוא תנאי שפה מקומי אליפטי ל-A הצמוד לעצמו. אנו מציידים את $\overline{Ell}(E)$ בטופולוגית C^1 בסמלים של האופרטורים, בטופולוגית C^0 באיברים חופשיים של האופרטורים, ובטופולוגית C^1 בתנאיי השפה.

הנתונים הטופולוגיים

מכל בעיית ערכי שפה (A, L) אנו מוציאים נתונים טופולוגיים המקודדים בתת-אגד ווקטורי $F = F(A, L)$ של E_∂, כאשר E_∂ הוא הצמצום של E לשפה ∂M. תת-האגד $F(A, L)$ תלוי רק ב- L ובצמצום של הסמל של A לשפה. כפי שאנו מראים בתזה, $F(A, L)$ מכיל את כל המידע אודות (A, L) הדרוש לנו כדי לחשב את הזרימה הספקטרלית ואת אינדקס המשפחה.

הזרימה הספקטרלית

נזכיר שלמסילה של אופרטורי Fredholm הצמודים לעצמם יש אינווריאנט אנליטי חשוב, הזרימה הספקטרלית, השווה למספר הערכים העצמיים החוצים את האפס (הנמנה עם סימנים מתאימים) במעבר מתחילת המסילה לסופה.

תהי $\gamma\colon [0,1] \to \overline{Ell}(E)$, $\gamma_t = (A_t, L_t)$, מסילה רציפה של בעיות ערכי שפה כך ש- $A_1 = gA_0g^*$ ו- $L_1 = gL_0$ עבור איזשהו אוטומורפיזם אוניטרי g של E. אז המשפחה החד-פרמטרית $F_t = F(A_t, L_t)$ של תת-אגדים של E_∂ מקיימת את התנאי $F_1 = gF_0$. נדביק F_1 ל-F_0 עם הפיתול הניתן על ידי g ונקבל אגד ווקטורי $\mathcal{F}(\gamma, g)$ מעל $\partial M \times S^1$.

התוצאה העיקרית של חלק V של התזה היא נוסחת הזרימה הספקטרלית:

משפט 1. *הזרימה הספקטרלית של* γ *שווה לערך של מחלקת* Chern *הראשונה של* $\mathcal{F}(\gamma, g)$ *על מחלקת ההומולוגיה היסודית של* $\partial M \times \mathrm{S}^1$,

$$\mathrm{sf}(\gamma) = c_1\big(\mathcal{F}(\gamma, g)\big)[\partial M \times \mathrm{S}^1].$$

בנוסף, אנו מראים שהזרימה הספקטרלית היא אינווריאנט אדיטיבי אוניברסלי של מסילות כאלה, אם נדרוש התאפסות על מסילות של אופרטורים הפיכים.

אינדקס המשפחה

בחלק VI אנו מכלילים את התוצאות של חלק V למשפחות של בעיות ערכי שפה התלויות בפרמטר הרץ במרחב קומפקטי כלשהו X. במקרה הזה, הזרימה הספקטרלית עם ערכים שלמים מוחלפת על ידי האינדקס האנליטי המקבל ערכים בחבורה האבלית $\mathrm{K}^1(X)$.

לכל משפחה $\gamma\colon x \mapsto (A_x, L_x)$ כזאת אנו מתאימים את האינדקס הטופולוגי $\mathrm{ind}_t(\gamma) \in \mathrm{K}^1(X)$ כדלהלן. המשפחה $F_x = F(A_x, L_x)$ של תת-אגדים של E_∂ מגדירה אגד ווקטורי $\mathcal{F}(\gamma)$ מעל $\partial M \times X$. הגורם הראשון ∂M הוא האיחוד הזר של רכיבי השפה, שכל אחד מהם הוא מעגל. בעזרת ההומומורפיזם הטבעי $\mathrm{K}^0(\mathrm{S}^1 \times X) \to \mathrm{K}^1(X)$ והסיכום מעל מרכיבי השפה, נקבל הומומורפיזם

$$\mathrm{Ind}_t\colon\ \mathrm{K}^0(\partial M \times X) \to \mathrm{K}^1(X).$$

נגדיר את האינדקס הטופולוגי של γ כערך של Ind_t על המחלקה של $\mathcal{F}(\gamma)$ ב- $\mathrm{K}^0(\partial M \times X)$,

$$\mathrm{ind}_t(\gamma) \coloneqq \mathrm{Ind}_t[\mathcal{F}(\gamma)] \in \mathrm{K}^1(X).$$

התוצאה הראשונה של חלק VI היא משפט אינדקס המשפחה:

משפט 2. *האינדקס האנליטי של* γ *שווה לאינדקס הטופולוגי שלו.*

התוצאה השניה של חלק VI היא האוניברסליות של האינדקס למשפחות של בעיות ערכי שפה כאלה. אנו מראים שחבורת Grothendieck של מחלקות הומוטופיה של משפחות כאלה מודולו תת-החבורה של משפחות הפיכות איזומורפית ל- $\mathrm{K}^1(X)$, כאשר האיזומורפיזם ניתן על ידי האינדקס. אנו מוכיחים תוצאות עוד יותר חזקות לגבי חבורות למחצה של משפחות כאלה, בלי לעבור לחבורת Grothendieck.